1 THE QUANTUM MULTIPOLE RADIATION

ALEXANDER S. SHUMOVSKY

Physics Department, Bilkent University
Bilkent, Ankara, Turkey

There are some points which are as dark as ever: But we have so much that it will be our own fault if we cannot get the rest.

-Sir Arthur Conan Doyl: The Second Stain

Contents

1.1 INTRODUCTION

Before we start to investigate that, let us try to realize what we do know, so as to make the most of it, and to separate the essential from the accidental.

-Sir Arthur Conan Doyl: The Priory School

Since the pioneering paper by Dirac [1], the formalism of quantum electrodynamics (QED) is based on the use of the photon creation and annihilation operators, forming a representation of the Weyl-Heisenberg algebra, and on the notion of the electromagnetic vacuum state (e.g., see [2, 3, 4]). As far as a denumerable set of Fock number states can be generated from the vacuum state by successive action of the creation operator, one can choose to interpret the electromagnetic vacuum as a "physical system" ready for support of any electromagnetic radiation.

It is not heretical to consider the electromagnetic vacuum as a "physical system". In fact, it manifests some physical properties and is responsible for a number of important effects. For example, the field amplitudes continue to oscillate in the vacuum state. These zero-point oscillations cause the spontaneous emission [1], the natural line breadth [5], the Lamb shift [6], the Casimir force between conductors [7] and the quantum beats [8]. It is also possible to generate quantum states of electromagnetic field in which the amplitude fluctuations are reduced below the symmetric quantum limit of zero-point oscillations in one quadrature component [9].

In spite of the great success of QED, there still is a number of unclear principal problems (e.g., see [10, 11, 12, 13, 14, 15]). Leaving aside the detailed discussion of foundations of QED, we shall concentrate here on the problems of localization of photons and quantum phase of electromagnetic radiation, which have attracted a great deal of interest.

The point is that the photon creation and annihilation operators are defined in QED as nonlocal objects. In other words, the photon number operator gives the total number of photons in the volume of quantization without specification of their space-time location [14, 15]. Moreover, it has been proven by Newton and Wigner [16] that no position operator can exist for the photon. There is a widespread belief that the maximum precise localization appears in the form of a wavefront [17]. At the same time, the specific fall-off of the photon energy density and of the photodetection rate can be interpreted as the photon localization in space [18, 19].

Perhaps, the most evident and bright example of the photon localization is provided by the photodetection process, when photon is transformed into electronic signal in the sensitive element of the detecting device [20]. Such a localization is usually described in the operational way (in terms of what can be measured by a macroscopic detector) through the use of the so-called configuration number operator, determining the number of photons in the cylindrical volume $\sigma c \Delta t$, where σ denotes the area of the sensitive element, c is the light velocity, and At is the detector exposition time [14, 20]. Another interesting examples are provided by the localization in photonic

crystals [21] and by the emission and absorption of radiation by atoms and molecules [22].

We now stress that, in the usual treatment of photon localization, the radiation field is considered as though it consist of the *plane waves of photons* [14, 15, 16, 17,201. In reality, the radiation emitted by the atomic transitions corresponds to *the multipole photons* [23] represented by the quantized spherical waves [2].

Although the classical plane and spherical waves are equivalent in the sense that they both form complete orthogonal sets of solutions of the homogeneous Helmholtz wave equation [24, 25], there is a strong qualitative difference between the two quantum representations. The plane waves of photons correspond to the running-wave solutions of the homogeneous Helmholtz wave equation in a large but finite cubic cavity with periodic boundary conditions [1, 2, 14, 15]. This choice of the boundary conditions corresponds to the translational symmetry of solutions and leads to the states of photons with given linear momentum [3, 10,111. In turn, the solution of the homogeneous Helmholtz wave equation in terms of spherical waves assumes the existence of a singular point, corresponding to an atom (source or absorber of radiation) whose size is small with respect to the wavelength [24, 25, 26]. In this case, the boundary conditions correspond to the rotational symmetry and lead to the states of photons with given angular momentum [2, 4, 27]. Since the components of linear and angular momenta do not commute, the two representations of quantum electromagnetic field correspond to the physical quantities which cannot be measured at once.

The simplest way to show the principal difference between the representations of plane and multipole photons is to compare the number of independent quantum operators (degrees of freedom), describing the monochromatic radiation field. In the case of plane waves of photons with given wave vector $\vec{k}$ (energy and linear momentum), there are only two independent creation or annihilation operators of photons with different polarization [2, 14, 15]. It is well known that QED interprets the polarization as given spin state of photons [4]. The spin of photon is known to be 1, so that there are three possible spin states. In the case of plane waves, projection of spin on the direction of $\vec{k}$ is forbidden due to the translational invariance and hence only two transversal polarizations are allowed [4].

In turn, the monochromatic multipole photons are described by the scalar wave number k (energy), parity (type of radiation either electric or magnetic), angular momentum $j = 1, 2, \cdot\,\cdot$, and projection $m = -j, \ldots, j$ [2, 26, 27]. This means that even in the simplest case of monochromatic dipole ($j = 1$) photons of either type, there are three independent creation or annihilation operators labeled by the index $m = 0$, fl. Thus, the representation of multipole photons has much physical properties in comparison with the plane waves of photons. For example, the third spin state is allowed in this case and therefore the quantum multipole radiation is specified by three different polarizations, two transversal and one longitudinal (with respect to the radial direction from the source) [27, 28]. In contrast to the plane waves of photons, the projection of spin is not a quantum number in the case of multipole

photons. Therefore, the polarization is not a global characteristics of the multipole radiation but changes with distance from the source [22].

Another very important difference between the plane and multipole photons consists in the character of zero-point oscillations of the field strengths [29]. We shall show here that, unlike the former case with spatially homogeneous zero-point oscillations, the multipole vacuum noise strongly depends on the distance from the singular point (atom). It is not an unexpected result. In fact, the zero-point oscillations reflect the structure of the electromagnetic vacuum state which, in turn, depends on the boundary conditions for the homogeneous Helmholtz wave equation [3]. Let us note in this connection that possible influence of an atom on the electromagnetic vacuum state in the absence of radiation has been discussed in QED for a long time (e.g., see [30, 31]). It should be stressed that the spatial inhomogeneity of the multipole vacuum noise can be very important for prognosis of experiments with trapped atoms [32] and single-atom laser [33], especially in the engineered entanglement in the atom-photon systems [32].

We now note that, in recent years, the entanglement has been recognized as one of the most fundamental features of quantum systems as well as an important tool of quantum communication and information processing (e.g., see [34]). One of the promising ways in the engineered entanglement is represented by the so-called two-photon polarization entanglement (see Sec. 12.14 in [14]). In this case, the cascade decay of an atomic transition leads to the creation of two entangled photons with different polarizations and different directions of propagation. Therefore, an adequate estimation of the vacuum noise in the atom-photon interactions seems to be of great importance.

While the simplified picture based on the model of plane waves of photons, neglecting the presence of sources and absorbers, is incapable of describing the photon localization, we show here that the use of the rich physical properties of multipole photons leads to an adequate description of localization in the atom-field interaction processes as well as in conventional photodetection. We note that the causal relation between the boundary conditions for the homogeneous Helmholtz wave equation and photon localization has been discussed recently in [19, 22, 29, 35].

The representation of multipole photons is also useful in the investigation of the quantum phase problem [36]. In the pioneering paper [1] on the quantization of electromagnetic field, Dirac first postulated the existence of a Hermitian phase operator defined by the polar decomposition of the annihilation operator and conjugated to the photon number operator. Later it was realized that the Dirac's phase operator cannot be considered as a properly defined Hermitian operator, describing the quantum phase properties of electromagnetic radiation (see, for review [14, 37, 38, 39, 40]. In particular, Susskind and Glogower [41] emphasized that the main difficulty in the correct definition of the phase operator arises because the spectrum of the number operator is bounded from below. An extension of the eigenvalue spectrum to negative values allows for the correct mathematical construction of the Hermitian phase operator [42, 43] while leads to non-physical states. An attempt to use the cosine and

sine of the phase operators rather than the quantum phase operator was discussed in [44].

A way to overcome the difficulties in the definition of the Hermitian phase operator has been proposed by Pegg and Bamett [40, 45]. Their method is based on a contraction of the infinite-dimensional Hilbert-Fock space of photon states $\mathcal{H}$. Within this method, the quantum phase variable is determined first in a finite s-dimensional sub-space of $\mathcal{H}$, where the polar decomposition is allowed. The formal limit $s \to \infty$ is taken only after the averages of the operators, describing the physical quantities, have been calculated. Let us stress that any restriction of dimension of the Hilbert-Fock space of photons is equivalent to an effective violation of the algebraic properties of the photon operators and therefore can lead to an inadequate picture of quantum fluctuations [46].

Perhaps, the most important result in the field of quantum phase problem was obtained by Mandel *et al* [47] within the framework of the operational approach. According to their analysis, ***there is no unique quantum phase variable, describing universally the measured phase properties of light.*** This very strong statement has obtained a totally convincing confirmation in a number of recent experiments [47, 48]. The results of the operational approach can be interpreted with the aid of the method based on the special quasiprobability distribution functions [49].

Generally speaking, the quantum phase variables can be divided into two classes. First of all, we have the pure operational phases which are completely determined by the scheme of measurement. This has no contradiction with the existence of an intrinsic quantum-dynamical variable responsible for the phase properties of light [50]. In addition, there might be some ***inherent*** quantum phases related to the quantum properties of photons. Since any photon can be specified by its energy, angular momentum and/or linear momentum, the inherent phase should be determined by either the angular or linear momenta, as the energy is a scalar. The former is connected with the spin states and hence, with the polarization of radiation field. The latter can lead to some "geometrical" phase, which, for example, can be measured as the phase difference between two plane waves emitted by one source in opposite directions.

It is well known that the angular momentum of a quantum mechanical system is specified by a representation of the $SU(2)$ algebra. If the corresponding enveloping algebra contains a uniquely defined scalar (the Casimir operator), the polar decomposition of the angular momentum can be obtained [51]. This polar decomposition determines a dual representation of the $SU(2)$ algebra expressed in terms of so-called phase states [51]. In particular, the Hermitian operator of the $SU(2)$ quantum phase can be constructed [51].

Although the angular momentum of quantum multipole radiation is well defined in terms of the multipole-photon operators of creation and annihilation, the direct polar decomposition of corresponding $SU(2)$ sub-algebra in the Weyl-Heisenberg algebra is impossible. The point is that this $SU(2)$ sub-algebra has no isotype representation [52]. This means that the Casimir operator (scalar) cannot be uniquely determined in the whole Hilbert-Fock space of photon states. Hence, the quantum phase of

the angular momentum of multipole photons cannot be determined by the method proposed in [51] valid for the quantum mechanical systems.

An approach focused on overcoming this difficulty was developed recently [36, 46, 53, 54]. The main idea, which seems to be a very natural one, is to consider the radiation of a given quantum source (atom or molecule) rather than a source-free electromagnetic field represented by the plane waves. Even in the classical picture, the multipole radiation can be determined completely only if the source functions, describing a local source at the origin, are known [25]. Within the quantum picture where the atom-field interaction is described in terms of the perturbation theory [26], we can take into account the source dependence of radiation using the conservation laws. In particular, the conservation of angular momentum in the process of radiation [26] permits us first to define the $SU(2)$ quantum phase of the atomic transition, following the method by Vourdas [51], an then to construct an operator complement of the atomic cosine and sine operators with respect to the integrals of motion in the whole atom-field system [36].

Many attempts have been made to define the quantum phase of light via the angular momentum (e.g., see [55] and references therein). The new element of our approach [36, 46, 53, 54] is that we determine the quantum phase of radiation via the quantum phase of the angular momentum of its source.

Let us stress that the electromagnetic vacuum state has no phase at all. This is the same as saying that the vacuum state is degenerated with respect to the phase or that the phase is distributed uniformly over the vacuum [14, 15]. The degeneration is taken off in the process of creation of the photon by atomic transition. Thus, it seems to be quite natural to assume that the inherent quantum phase of photons is generated by the source [36, 46]. Definitely, this is not an unusual assumption. Actually, the classical amplitudes of the multipole field are completely determined by the source functions, describing the charge density, current density and magnetization [25]. Hence, the multipole photon operators, which are obtained by the quantization of classical amplitudes [1, 2], are also specified by the source [56].

We also note that, in contrast to the Pegg-Bamett formalism [45], we consider an extended space of states, including the Hilbert-Fock state of photons as well as the space of atomic states [36, 46, 53, 54]. The quantum phase of radiation is defined, in this case, by mapping of corresponding operators from the atomic space of states to the whole Hilbert-Fock space of photons. This procedure does not lead to any violation of the algebraic properties of multipole photons and therefore gives an adequate picture of quantum phase fluctuations [46].

We provide here a review of investigations of the photon localization and quantum phase problems based on the use of the representation of multipole photons. Section 2 contains a general consideration of the field quantization. In particular, we compare the zero-point oscillations of the plane and multipole waves of photons and show that the vacuum noise is concentrated in some vicinity of atoms. In Section 3 we discuss the atom-field interaction leading to the multipole radiation and consider the $SU(2)$ quantum phase representation of atomic variables. Here we also discuss a connection between the SU(2) quantum phase states and entanglement phenomenon. In Sec-

tion 4 we describe the quantum phase of multipole radiation caused by the angular momentum conservation in the process of radiation. We compare this approach with the Pegg-Bamett formalism and with Mandel's operational approach. In Section 5 we consider the quantum polarization properties of multipole radiation. Then, in Section 6, we discuss the photon localization, quantum measurements and causality. To simplify the reading, we supplement each section by a brief resume. A general conclusion and the implications of this work are presented in Section 7.

1.2 QUANTUM MULTIPOLE FIELD

We must not think of the things that we could do with, but only of the things we can't do without.

-Jerome K. Jerome: Three Men in a Boat.

1.2.1 Classical electromagnetic field

An arbitrary free classical electromagnetic field is described by the vector potential $\vec{\mathcal{A}}(\vec{r})$ which obeys the wave equation [14, 24, 25]

$$\nabla^2 \vec{\mathcal{A}} - \frac{1}{c^2}\frac{\partial^2 \vec{\mathcal{A}}}{\partial t^2} = 0 \tag{1.1}$$

and Coulomb gauge condition

$$\vec{\nabla} \cdot \vec{\mathcal{A}} = 0. \tag{1.2}$$

The field strengths are then defined as follows

$$\vec{\mathcal{E}} = -\frac{1}{c}\frac{\partial \vec{\mathcal{A}}}{\partial t}, \qquad \vec{\mathcal{B}} = \vec{\nabla} \times \vec{\mathcal{A}}. \tag{1.3}$$

The equation (1) can be solved by separation of variables [24]

$$\vec{\mathcal{A}}(\vec{r}, t) = \sum_{\ell} q_\ell(t)\vec{u}_\ell(\vec{r}). \tag{1.4}$$

Employing (1) then gives the homogeneous Helmholtz wave equations of the form

$$\begin{aligned} d^2 q_\ell / dt^2 \; + \; \omega_\ell^2 q_\ell \; &= \; 0 \; , \\ \nabla^2 \vec{u}_\ell \; + \frac{\omega_{\ell \cdot \iota}^2}{c^2 \frac{\iota}{n}} \vec{u}_\ell \; &= \; 0, \end{aligned} \tag{1.5}$$

where ω_ℓ are some constants, arising from the separation of variables [24]. Solution of the first equation in (5) gives the harmonic time dependence

$$q_\ell = \exp(\pm i\omega_\ell t).$$

Due to the harmonic time dependence in (4), it is customary to represent the vector potential in terms of the positive and negative frequency parts:

$$\vec{A}(\vec{r}) = \vec{A}(\vec{r}) + \vec{A}^*(\vec{r}), \tag{1.6}$$

where $\vec{A} \sim \exp(-i\omega t)$.

The energy density of the field is

$$W(\vec{r}) = \frac{1}{16\pi}[\vec{\mathcal{E}}^*(\vec{r}) \cdot \vec{\mathcal{E}}(\vec{r}) + \vec{B}^*(\vec{r}) \cdot \vec{B}(\vec{r})]. \tag{1.7}$$

In turn, the flux of energy is given by the real part of the complex Poynting vector

$$\vec{S}(\vec{r}) = \frac{1}{8\pi}\vec{E}(\vec{r}) \times \vec{B}^*(\vec{r}), \tag{1.8}$$

where, according to (3),

$$\vec{E}(\vec{r}) = -ik\vec{A}(\vec{r}), \qquad \vec{B}(\vec{r}) = \text{a x } \vec{A}(\vec{r}).$$

The angular momentum density of the field has the form [25]

$$\vec{M}(\vec{r}) = \frac{1}{4\pi c}\vec{r} \times [\vec{\mathcal{E}}(\vec{r}) \times \vec{B}(\vec{r})]. \tag{1.9}$$

One of the possible solutions of the first equation in (5), corresponding to the plane waves, traveling along the z-axis and having the same amplitude and phase everywhere [24], has the form [14, 24, 25]

$$\vec{u}_\ell(\vec{r}) = \sum_{\text{e}} \sum_{\sigma=x,y} \vec{e}_{\ell\sigma} e^{i\vec{k}_\ell \cdot \vec{r}} a_{\ell\sigma} + C.c. \tag{1.10}$$

Here $a_{\ell\sigma}$ are the complex field amplitudes, $\vec{e}_{x,y}$ are the unit vectors of polarization which, due to the Coulomb condition (2), obey the relation

$$ve \quad \vec{e}_{x,y} \cdot \vec{k}_\ell = 0 \tag{1.11}$$

and $k_\ell^2 = \omega_\ell^2/c^2$. Employing (3), (6) and (10) then gives

$$\begin{aligned} E_x(\vec{r}) &= i\sum_k kA_{kx}(\vec{r}) = B_y(\vec{r}), \\ E_y(\vec{r}) &= i\sum_k kA_{ky}(\vec{r}) = -B_x(\vec{r}). \end{aligned} \tag{1.12}$$

To simplify the notations we omit here the index ℓ. Here, according to (lo), we have

$$A(\vec{r}) = \sum_{k} \gamma_k \sum_{\sigma=x,y} \vec{e}_{k\sigma} e^{i\vec{k}\cdot\vec{r}} a_{k\sigma} e^{-i\omega t}, \tag{1.13}$$

where γ_k is the normalization factor. Another possible solution of the homogeneous Helmholtz wave equation (5) convenient for electromagnetic boundary-value problems possessing spherical symmetry properties is provided by the spherical waves [24, 25]. In this case, it is supposed that there is a singular point at the origin, corresponding to a localized source distribution or to an absorber (in the case of incoming spherical waves).

In the spherical coordinates

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

the second equation in (5) takes the form [24]

$$\frac{\partial^2 u_\ell}{\partial r^2} + \frac{2}{r}\frac{\partial u_\ell}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_\ell}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u_\ell}{\partial\phi^2} + \frac{\omega_\ell^2}{c^2}u_\ell = 0. \tag{1.14}$$

Corresponding solution can be found by the separation of variables

$$u = R(r)\Theta(\theta)\Phi(\phi),$$

in the mode function in (5), which yields the following set of ordinary differential equations [24]

$$\begin{aligned} \frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[\frac{\omega^2}{c^2} - j(j+1)\right]R &= 0, \\ \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[j(j+1) - \frac{m^2}{\sin^2\theta}\right]\Theta &= 0, \\ \frac{d^2\Phi}{d\phi^2} + m^2\Phi &= 0. \end{aligned} \tag{1.15}$$

The solution of these equations is represented by certain combinations of spherical Bessel or Hankel functions and spherical harmonics [24, 25, 26].

To establish a contact with the quantum picture, consider the so-called helicity basis [27]

$$\vec{\chi}_\pm = \mp\frac{\vec{e}_x \pm i\vec{e}_y}{\sqrt{2}}, \qquad \vec{\chi}_0 = \vec{e}_z. \tag{1.16}$$

It is clear that $\{\vec{\chi}_\mu\}$ formally coincide with the three states of spin 1 of a photon. Therefore, one can choose to interpret $\vec{\chi}_\pm$ as the unit vectors of circular polarization

with either positive or negative helicity, while $\vec{\chi}_0$ gives the linear polarization in the z-direction [27]. We note here that to within the sign at $\vec{\chi}_\pm$ the helicity basis (16) coincides with the so-called polarization basis usually used in optics [57].

In the basis (16), any vector $\vec{A}$ can be expanded as follows

$$\vec{A} = \sum_{\mu=-1}^{1} (-1)^\mu \vec{\chi}_{-\mu} A_\mu.$$

In this basis, for the positive-frequency part of the vector potential in (6) we get [2, 24, 25, 26, 27]

$$\vec{A}_\lambda(\vec{r}) = \sum_k \sum_\mu \sum_j \sum_{m=-j}^{j} (-1)^\mu \vec{\chi}_{-\mu} V_{\lambda kjm\mu}(\vec{r}) a_{\lambda kjm} e^{-i\omega t}. \tag{1.17}$$

Here $\lambda = E, M$ denotes the type of radiation, either electric or magnetic, index j takes the values $1, 2, \ldots$ and index m = -j, . . . , j. The complex field amplitudes are defined in terms of the source functions, describing the local distribution of current and intrinsic magnetization [25]. The mode functions in (17) can be represented in the following form [2, 26, 27]

$$\begin{aligned} V_{Ekjm\mu} &= \gamma_{Ekj}[\sqrt{j} f_{j+1}(kr)\langle 1, j+1, \mu, m-\mu | jm\rangle Y_{j+1,m-\mu}(\theta\phi) \\ &- \sqrt{j+1} f_{j-1}(kr)\langle 1, j-1, \mu, m-\mu | jm\rangle Y_{j-1,m-\mu}(\theta,\phi)], \\ V_{Mkjm\mu} &= \gamma_{Mkj} f_j(kr)\langle 1, j, \mu, m-\mu | jm\rangle Y_{jm}(\theta,\phi). \end{aligned} \tag{1.18}$$

Here $\gamma_{\lambda kj}$ is the normalization constant, $\langle \ldots | jm\rangle$ denotes the Clebsch-Gordon coefficient, and $Y_{\ell m}$ is the spherical harmonics. The radial contribution into the mode functions (18) depends on the boundary conditions as follows [24]

$$f_\ell(kr) = \begin{cases} h_\ell^{(1)}(kr), & \text{outgoing spherical wave} \\ h_\ell^{(2)}(kr), & \text{incoming spherical wave} \\ j_\ell(kr), & \text{standing spherical wave} \end{cases} \tag{1.19}$$

where $h_\ell^{(1,2)}$ denotes the spherical Hankel function of the first and second kind respectively and j_ℓ is the spherical Bessel function [24, 25].

Unlike the case of plane waves of photons, the multipole field (18) propagates as a uniformly expanding spherical shell rather than propagates along given direction of k. Instead of the symmetry relations (12), for the spherical waves of photons we get the following reciprocity relations [2, 27]

$$\begin{aligned} \vec{E}_{Ekjm} &= \vec{B}_{Mkjm} = ik\vec{A}_{Ekjm}, \\ \vec{E}_{Mkjm} &= -\vec{B}_{Ekjm} = lk\vec{A}_{Mkjm}. \end{aligned} \tag{1.20}$$

1.2.2 Quantum electromagnetic field

The canonical quantization of the field has introduced by Dirac [1] (also see Refs. [2, 3, 4, 10, 11, 14, 15, 26, 27]) is provided by the substitution of the photon operators, forming a representation of the Weyl-Heisenberg algebra, into the expression for the vector potential instead of the complex field amplitudes. For example, in the case of plane waves, described by the positive-frequency part of the vector potential (13), we get the following operator construction

$$A(\vec{r}) = \sum_{k} \sum_{\sigma=x,y} \sqrt{\frac{2\pi\hbar c}{kV}} \vec{e}_{k\sigma} e^{i\vec{k}\cdot\vec{r}} a_{k\sigma}, \tag{1.21}$$

where V is the volume of quantization which is supposed to be a large cubic box with periodical boundary conditions. Here the harmonic time dependence is included into the photon operators which obey the commutation relations

$$[a_{k\sigma}, a^{+}_{k'\sigma'}] = \delta_{kk'}\delta_{\sigma\sigma'}. \tag{1.22}$$

Due to the translational symmetry along the z-direction, the plane waves of photons, described by (21) and (22), correspond to the states of the radiation field with given linear momentum

$$\vec{P} = \sum_{k\sigma} \hbar\vec{k} a^{+}_{k\sigma} a_{k\sigma},$$

where $\vec{k} = k\vec{e}_z$.

The multipole electromagnetic field ***"can be quantized in much the same way as plane waves"*** [2]. We have to subject the complex field amplitudes in the expansion (17) to the Weyl-Heisenberg commutation relations of the form

$$[a_{\lambda kjm}, a^{+}_{\lambda' k' j' m'}] = \delta_{\lambda\lambda'}\delta_{kk'}\delta_{jj'}\delta_{mm'}, \tag{1.23}$$

Then, the positive-frequency part of the operator vector potential of the multipole radiation of a given type λ takes the form [2, 27]

$$\vec{A}_{\lambda}(\vec{r}) = \sum_{k} \sum_{\mu} \sum_{j} \sum_{m=-j}^{j} (-1)^{\mu} \vec{\chi}_{-\mu} V_{\lambda kjm\mu}(\vec{r}) a_{\lambda kjm}, \tag{1.24}$$

where the harmonic time dependence is again included into the definition of the photon operators of creation and annihilation. In the case of standing waves of photons in an ideal spherical cavity of volume V, the normalization factors in (18) take the form [2, 27]

$$\gamma_{Ekj} = \sqrt{\frac{2\pi\hbar c}{kV(2j+1)}}$$

$$\gamma_{Mkj} = \sqrt{\frac{2\pi\hbar c}{kV}}.$$

Within the quantum picture, the Clebsch-Gordon coefficients in (18) represent the vector addition of the spin and orbital part of the total angular momentum of the field [2]. The indexes j and m in (23) and (24) correspond to the angular momentum and projection of angular momentum on the quantization axis. The electric-type multipole radiation is interpreted as having the parity of state $(-1)^{j+1}$, while the magnetic-type multipole radiation is specified by the states with parity $(-1)^{j}$. Due to the spherical symmetry of solutions (17)-(19), the representation of spherical waves of photons (23) ,(24) corresponds to the states of quantum multipole field with given angular momentum. Since the components of the linear and angular momenta do not commute with each other, the two representations (21), (22) and (23), (24) are different in principle. They correspond to the physical observables which cannot be measured simultaneously.

For both the plane and multipole waves of photons, the vacuum state can be defined by the stability condition in the same way [3, 4]:

$$\begin{aligned} \forall k,\sigma \qquad & a_{k\sigma}|0\rangle = 0, \\ \forall \lambda,k,j,m \qquad & a_{\lambda kjm}|0\rangle = 0. \end{aligned} \tag{1.25}$$

Then, the corresponding Fock number states are defined as follows

$$\begin{aligned} |n_{k\sigma}\rangle &= \frac{(a^{+}_{k\sigma})^{n}_{k\sigma}}{\sqrt{n_{k\sigma}!}}, \\ |n_{\lambda kjm}\rangle &= \frac{a^{+}_{\lambda kjm})^{n}_{\lambda kjm}}{\sqrt{n_{\lambda kjm}}}, \end{aligned} \tag{1.26}$$

where n... ≥ 0 is an integer.

As can be seen from the equations (21), (22), and (23), (24), there is an essential difference between the representations of plane and multipole waves of photons. In particular, a monochromatic plane wave of photons is specified by only two different quantum numbers and $\sigma = x, y$, describing the linear polarization in Cartesian coordinates. In turn, the monochromatic multipole photons are described by much more quantum numbers. Even in the simplest case of the electric dipole radiation when $\lambda = E$ and $j = 1$, we have three different states of multipole photons in (23) with m = 0, fl. Besides that, the plane waves of photons have the same polarization σ everywhere, while tbe states of multipole photons have given m. It is seen from (24) that, in this case, the polarization described by the spin index μ can have different values at different distances from the singular point. More detailed discussion of the polarization properties of the multipole radiation we postpone till Section 5.

A deeper difference of the two representations can be traced in the properties of the zero-point oscillations. In fact, the energy operators obtained by quantization of

(7) in the plane and spherical wave representations have the form [26, 27, 58, 59]

$$H_{(plane)} = \sum_{k,\sigma} \hbar\omega_k (a^+_{k\sigma} a_{k\sigma} + 1/2),$$

$$H_{(multi)} = \sum_{k} \hbar\omega_k \sum_{\lambda,j,m} (a^+_{\lambda jm} a_{\lambda jm} + 1/2).$$

Thus, the energy of the vacuum state are

$$H^{(vac)}_{(plane)} = \sum_{k,\upsilon} \hbar\omega_k/2 = \sum_{k} \hbar\omega_k,$$

$$H^{(vac)}_{(multi)} = \sum_{k} \hbar\omega_k \sum_{\lambda,j,m} 1/2 = \sum_{k} \hbar\omega_k \left(\sum_{j} (2j+1) \right). \qquad (1.27)$$

Due to the definition of k, both expressions give an infinite energy and, at first sight, cannot be compared with each other. In fact, this infinity is inessential because of the following reason. The contribution of zero-point oscillations can be observed only via measurement which implies an averaging of physical quantities over a finite "volume of detection" and exposition time of detector. Such an averaging plays a part of filtration leading to a selection of a certain finite transmission frequency band [58]. It is then seen that, even if the filtration process leads to separation of the dipole photons only, the second term in (27) exceeds the first one in three times. From the physical point of view, this result is caused by the more number of quantum degrees of freedom in the case of multipole photons.

Much more interesting and important result can be obtained from the consideration of the spatial properties of the vacuum fluctuations. The simplest example is provided by the calculation of vacuum average of the squared electric field strength [58, 59]

$$W(\vec{r}) = \langle 0|\vec{\mathcal{E}} \cdot \vec{\mathcal{E}}|0\rangle = k^2 \langle 0|\vec{A} \cdot \vec{A}|0\rangle,$$

obtained from (6) by the canonical quantization of the field. It follows from the definition of the vacuum state (25) that this expression can be put into the form

$$W(\vec{r}) = k^2 \langle 0|[\vec{A}, \vec{A}^+]|0\rangle = k^2 [\vec{A}, \vec{A}^+]$$

independent of the type of representation. Consider first the monochromatic plane waves of photons. Using (21) together with the commutation relations (22), we get

$$W^{(plane)}(\vec{r}) = W^{(plane)} = \sum_{k,\sigma} \frac{\hbar\omega_k}{2V}, \qquad \omega_k = kc. \qquad (1.28)$$

Here V has the same meaning as in (21). Thus, the zero-point fluctuations of the electric field strength of plane waves have the same magnitude at any space point. By construction, (28) describes the zero-point fluctuations ***in empty space.***

In turn, employing the representation (23), (24) then gives for (27) in the case of multipole photons the following

$$W^{(multi)}(\vec{r}) = k^2 \sum_{\lambda=E,M} \sum_{\mu} \sum_{k,j,m} |V_{\lambda kjm\mu}(\vec{r})|^2. \tag{1.29}$$

It is seen from the definition of the mode functions (18) and (19) that, in contrast to (28), the zero-point oscillations of the electric field strength of multipole photons manifest the *spatial inhomogeneity.*

For simplicity, we can compare the monochromatic contributions into (28) and (29) at the same k and V. Then

$$W_k^{(plane)} = \sum_{\sigma=1,2} \frac{\hbar kc}{2V}$$

and

$$W_k^{(multi)}(\vec{r}) = \sum_{jm\mu} \frac{2\pi\hbar kc}{V}$$

$$\times \left[\left| \sqrt{\frac{j}{2j+1}} f_{j+1}(kr) \langle 1, j+1, \mu, m-\mu | 1m \rangle Y_{j+1,m-\mu} \right.\right.$$

$$\left. - \sqrt{\frac{j+1}{2j+1}} f_{j-1}(kr) \langle 1, j-1, \mu, m-\mu | 1m \rangle Y_{j-1,m-\mu} \right|^2$$

$$\left.\left. + \left| f_j(kr) \langle 1, j, \mu, m-\mu | 1m \rangle Y_{j,m-\mu} \right|^2 \right] .$$

Since $Y_{\ell,m-\mu} \sim e^{i(m-\mu)\phi}$, this form is independent of the azimuthal angle ϕ. Moreover, it is a straightforward matter to arrive at the conclusion that

$$W_k^{(multi)}(\vec{r}) = W_k^{(multi)}(r)$$

(see discussion in Section 5.3).

Consider first the case of standing spherical waves in an ideal spherical cavity when, according to (19),

$$f_\ell(kr) = \sqrt{\frac{\pi}{2kr}} J_{\ell+1/2}(kr).$$

We stress here that, in the quantum theory of radiation, exactly the standing spherical waves are usually considered [2, 27]. Unlike the outgoing and incoming spherical waves of photons, this choice does not lead to the divergence of the vector potential at $r \to 0$. Taking into account the properties of Bessel functions $J_j(x)$, it is easily seen that the principal contribution into $W^{(multi)}$ in vicinity of the singular point (atom) comes from $J_{1/2}(kr)$, corresponding to the electric dipole radiation. The

radial dependence of $W^{(multi)}$ at fixed k and $j = 1$ is shown in Fig. 1. It is seen that the vacuum fluctuations are concentrated near atom where their level can strongly exceed that calculated within the framework of the model of plane waves.

The case of outgoing and incoming spherical waves can be examined under the standard assumption that the atom located at the origin has a finite size which permits us to avoid the divergence at $kr \to 0$. It is seen that, in some small vicinity of the atom, the zero-point oscillations, corresponding to the multipole field in an infinite space, strongly exceed those in the ideal spherical cavity (see Fig. 2).

It should be stressed that the above results have been obtained under the only assumption that the atom exists at the origin, no matter whether we use it as an emitter or absorber of radiation. In other words, the spatial inhomogeneity of the zero-point oscillations in (29) reflects the existence of the singular point which, in fact, is the boundary condition for the homogeneous Helmholtz wave equation (5). It is possible to say that the electromagnetic vacuum state "feels" the presence of an atom at the origin and is ready to support any radiation (with all possible λ, j and m) either outgoing or incoming. This is not an astonishing result. The influence of the electromagnetic vacuum state by the presence of an atom has been discussed in quantum electrodynamics for a long time [7, 30, 31]. The new result here is that the zero-point oscillations are concentrated in some vicinity of atoms where their level can exceed the standard level (28) which is usually considered.

The point is that the zero-point oscillations are responsible for the so-called shot noise [14, 15], determining the quantum limit of uncertainty in different optical measurements. The above result shows that the presence of an atom causes the increase of shot noise and hence a deterioration of the quantum limit of precision of measurements, at least, in some vicinity of the atom [22, 29]. We discuss this effect in more details in Section 6.

1.2.3 Resume

(i) Although the monochromatic plane waves of photons are described by only two quantum numbers, specifying the polarization, the monochromatic multipole waves of photons have much more quantum degrees of freedom: the type of radiation (parity) $\lambda = E, M$ and the angular momentum $j \geq 1$ and its projection $m = -j, \ldots, j$.

(ii) The zero-point oscillations of the energy density of plane waves of photons have the same magnitude everywhere. In contrast, those calculated in the presence of a singular point (source or absorber) manifest spatial inhomogeneity. Precisely, the vacuum noise is concentrated in some vicinity of the singular point.

1.3 ATOM-FIELD INTERACTION

"Well! ***I've often seen a cat without a grin", thought Alice; "but a grin without a cat! It's the most curious thing I ever saw in all my life".***

-Lewis Carroll: Alice's Adventures in Wonderland

1.3.1 Multipole Jaynes-Cummings model

In the previous Section, the classical and quantum electromagnetic fields were considered as absolutely free sourceless objects. This picture follows from the existence of nontrivial solutions of the homogeneous Hehnholtz wave equation (5). In some textbooks, this mathematical fact is interpreted as the claim of the following type: ***the electromagneticfield can exist in the absence of any charge (e.g., see*** [60]). At the same time, as we know, no one has ever observed photons that had not been created by a source. According to the quantum picture, the electromagnetic vacuum state contains unborn photons of all possible types. They are extracted from this state in the form of over-vacuum excitations (waves) in the process of source-photon interaction, leading to the photon generation. The observable properties of real photons are governed by this interaction, which causes the success of conventional [23, 61] and correlation [62, 63] spectroscopy. Rephrasing Lewis Carroll's "Alice's Adventures in Wonderland", it is possible to say that the source is similar to Cheshire Cat, creating grin which propagates in space-time (see Fig. 3).

The simplest quantum source of photons is the atomic transition, creating, due to the selection rules, multipole photons. The simplest model of the interaction of an atom with the electromagnetic radiation is associated with the notion of so-called two-level atom [64]. In fact, this model originates from the famous study of radiation kinetics by Einstein [65]. With the development of laser, the notion of two-level atom entered firmly into the practice of quantum optics. The fact is that, using lasers as sources of electromagnetic radiation, one can act on the atom with field having frequency very close to the transition frequency between any pair of levels. In this case, the influence of the other levels can be ignored, and one can restrict the consideration to a two-level atom (in general, an atom with a finite number of levels) [64]. On the other hand, the use of high-quality cavities has the consequence that an atom in such a cavity interacts with only one or very few modes of the field quantized in the volume of the cavity [32, 33, 66].

The branch of quantum optics studying the processes of interaction of one or a few atoms with the quantized cavity modes is usually called cavity quantum electrodynamics (cavity QED). The theoretical concepts of cavity QED are based in the first place on investigation of the Jaynes-Cummings model [67] and its generalizations (for a review, see [68]). The reason for this is that the model describes fairly well the physical processes under consideration and at the same time admits an exact solution.

In the usual formulation of the Jaynes-Cummings model, the atom is considered as though it consist of two non-degenerated levels [67] . In contrast, the radiative transitions in real atoms occur between the states with given angular quantum numbers $|j, \mathrm{m}) \to |j', \mathrm{m}')$ such that $j > j' \geq 0$ [23, 26, 61]. This means that, at least the upper level, is degenerated with respect to the quantum number $m\,(-j \leq m \leq j)$. For example, in the simplest case of the electric dipole transition between the states

$|j = 1, m = 0, fl)$ and $|j' = 0, m' = 0)$, the excited state is triple degenerated (see Fig. 4). The corresponding generalization of the Jaynes-Cummings model has been discussed in [36, 53]. Similar models have been considered in different problems of interaction of quantum light and matter [69].

Interaction between a single atom and radiation field is usually considered within the framework of perturbation theory using the following Hamiltonian [26, 64]

$$H = H_0 + \frac{1}{2m_e}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2, \tag{1.30}$$

where H_0 describes the unperturbed atom and field and the rest is written for the interaction between a single spinless electron with charge e, mass m_e, and momentum p and free electromagnetic field described by the vector potential $\vec{A}$. Following [36, 53], consider a two-level atom with the electric dipole transition between the triple-degenerated excited atomic state with $j = 1$ and nondegenerated ground state with j' = 0. The atom is supposed to be located at the center of an ideal spherical cavity. The coupling constant of the atom-field interaction can be found by calculating the matrix element [26, 27]

$$-\frac{e}{2m_e c}\langle 0,0|\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}|1,m\rangle = ik_0\langle 0,0|\vec{d}\cdot\vec{A}|1,m\rangle, \tag{1.31}$$

obtained from (30). The A^2 term is excluded due to the use of the so-called rotating-wave approximation [64]. Here

$$\vec{d} = e\vec{r}$$

is the dipole moment and $\vec{A}(\vec{r})$ is the operator vector potential (24) with the radial dependence of the mode functions (18) described by $f_\ell(kr) = j_\ell(kr)$ in (19) due to the choice of the boundary conditions.

Assuming the central symmetry of atomic field and taking into account the fact that the spin state of an atom does not change under the electric dipole transition [27, 61], we can represent the atomic states in (31) as follows

$$|1,m\rangle = \mathcal{R}_1(kr)Y_{1m}(\theta,\phi), \qquad |0,0\rangle = \mathcal{R}_0(kr)Y_{00}(\theta,\phi),$$

where $\mathcal{R}_\ell$ is the radial part of the atomic wave function. Then, representing the dipole moment $\vec{d}$ in the helicity basis (16) and carrying out the calculations of integrals in (3 1) over the atomic volume, we get

$$\forall m \;\; g \;\equiv k_0\langle 1,m|\vec{d}\cdot\vec{A}|0,0\rangle = k_o D\sqrt{\frac{3\hbar c}{10kR}}, \tag{1.32}$$

where

$$D = e\int_0^{r_a} dr r^3 f_1(kr) f_0(kr)$$

is the effective dipole factor. Here R and r_a denote the cavity and atomic radii respectively and k is the wave number, describing the cavity field.

Taking into account the explicit form of spherical Bessel functions [70]

$$j_0(kr) = \frac{\sin kr}{kr},$$
$$j_2(kr) = \left(3 - \frac{(kr)^2}{(kr)^3}\right)\sin kr - \frac{3\cos kr}{(kr)^2}, \qquad (1.33)$$

we note that, due to the structure of the mode functions (18), all other radial function do not contribute into (24). Assuming that the atom is a point-like object (in fact, very small with respect to the wavelength of radiation field), we get

$$\lim_{kr\to 0} j_0(kr) = 1, \qquad \lim_{kr\to 0} j_2(kr) = 0.$$

Using the properties of the Clebsch-Gordon coefficients [71] and spherical harmonics [70], for the mode functions (18) in this limit we get

$$V_{Ek1m}(0) \sim -\delta_{m\mu}.$$

Inserting this into (24) we obtain

$$A_{Ek1}(0) = -\sqrt{\frac{\hbar c}{3kV}} \sum_{m=-1}^{1} (-1)^{\mu} \vec{\chi}_{-\mu} a_{Ek1m} \delta_{m\mu}.$$

This means that the electric dipole transition $|1, m\rangle \to |0, 0\rangle$ creates a photon with spin state (polarization) $\mu = m$. However, the picture of the polarization changes with the distance from the atom due to the position dependence of the mode functions (18).

Thus, the Jaynes-Cummings Hamiltonian for the electric dipole transition can be written as follows [36, 53]

$$H = H_0 + H_{int},$$
$$H_0 = \sum_{m=-1}^{1} \{\omega a^{+}_{E1m} a_{E1m} + \omega_0 R_{mm}\},$$
$$H_{int} = g \sum_{m=-1}^{1} \{R_{mg} a_{E1m} + a^{+}_{E1m} R_{gm}\}. \qquad (1.34)$$

Here $\omega_0 = \hbar k_0 c$ and w = $\hbar kc$ are the energies of the atomic transition and cavity field respectively and the atomic operators are defined as follows

$$R_{mg} = |1, m\rangle\langle 0, 0|, \qquad R_{mm'} = |1, m\rangle\langle 1, m'|. \qquad (1.35)$$

The first term in (34) describes the energy of the free cavity field and atom, while the second term gives the energy either of the transition $|1, m) \to |0, 0)$ with generation of the multipole photon or of the transition $|0, 0) \to (1, m)$ accompanied by the absorption of corresponding electric dipole photon.

The generalizations of the Jaynes-Cummings model (34) on the case of quadrupole and other high-order multipole transitions can be constructed in the same way.

1.3.2 The $SU(2)$ atomic phase states

In the model Hamiltonian (34), the excited atomic state is specified by the following three orthogonal states

$$|1,1\rangle, \qquad |1,0\rangle, \qquad |1,-1\rangle. \tag{1.36}$$

In this basis, we can construct a representation of the $SU(2)$ algebra with the following generators (see [53, 54])

$$\begin{aligned} J_z &= R_{++} - R_{--}, \\ J_+ &= \sqrt{2}(R_{+0} + R_{0-}), \\ J_- &= \sqrt{2}(R_{0+} + R_{-}), \end{aligned} \tag{1.37}$$

which obey the standard commutation relations

$$[J_+, J_-] = 2J_z, \qquad [J_z, J_\pm] = \pm J_\pm. \tag{1.38}$$

The enveloping algebra of (37), (38) contains the uniquely defined Casimir operator

$$\mathbf{J}^2 = 2\sum_{m=-1}^{1} R_{mm} \equiv 2 \times \mathbf{1}, \tag{1.39}$$

where

$$\mathbf{1} = |1,1\rangle\langle 1,1| + |1,0\rangle\langle 1,0| + |1,-1\rangle\langle 1,-1|$$

is the unit operator in the space spanned by the basis (36).

The existence of (39) permits us to use the method, proposed by Vourdas [51], to construct the dual representation of the $SU(2)$ algebra (37), (38). Following [51], we represent the lowering and rising operators in (37) as follows

$$J_+ = J_r \varepsilon, \qquad J_- = \varepsilon^+ J_r, \tag{1.40}$$

where J_r is the Hermitian "radial" operator and ϵ is the unitary

$$\varepsilon\varepsilon^+ = \mathbf{1}$$

"exponential of the phase" operator. The terminology here is borrowed from complex calculus. It is clear that the phase variable here describes the azimuth of the angular momentum of the excited atomic state. The equations (40) can now be done in a straightforward manner to yield

$$\begin{aligned} J_r &= \sqrt{2}(1 - R_{--}), \\ \varepsilon &= R_{+0} + R_{0-} + e^{i\psi} R_{-+}, \end{aligned} \tag{1.41}$$

where ψ is an arbitrary real parameter describing the so-called atomic reference phase [53, 54]. It is clear that ϵ is a Coxeter-type operator [72] because

$$\varepsilon^3 = e^{i\psi} \times 1.$$

In analogy to complex calculus, we can now define the cosine and sine of the atomic $SU(2)$ phase operators [36]

$$C_a = (\mathrm{E} + \varepsilon^+)/2, \qquad S_a = (\varepsilon - \varepsilon^+)/2i \tag{1.42}$$

such that

$$C_a^2 + S_a^2 = 1 \tag{1.43}$$

and

$$[C_a, S_a] = 0. \tag{1.44}$$

Following [51], we now introduce the dual representation of the $SU(2)$ algebra (37), (38). Consider first the eigenstates of the exponential operator in (41)

$$\varepsilon|\phi_m\rangle = e^{i\phi_m}|\phi_m\rangle. \tag{1.45}$$

It is a straightforward matter to arrive at the relations [46]

$$|\phi_m\rangle = \frac{1}{\sqrt{3}} \sum_{m'=-1}^{1} e^{-im'\phi_m}|1, m\rangle, \qquad \phi_m = \frac{2m\pi - \psi}{3}, \tag{1.46}$$

where m takes the values 0 and ± 1 as above. It is easily seen that the so-called "phase states" [51] (45) determine the basis dual to (36) [46]. In particular

$$\sum_{m=-1}^{1} |\phi_m\rangle\langle\phi_m| = 1.$$

Then, the atomic $SU(2)$ quantum phase operator can be defined as follows

$$\hat{\phi} = \sum_{m=-1}^{1} \phi_m |\phi_m\rangle\langle\phi_m| = -\frac{\psi}{3}1 - \frac{2i\pi}{3\sqrt{3}}(e^{-i\psi/3}\epsilon - e^{i\psi/3}\epsilon^+). \tag{1.47}$$

In turn, the cosine and sine operators (42) can be represented as the following functions

$$C_a = \cos\hat{\phi}, \qquad S_a = \sin\hat{\phi}$$

of the operator (47). Then, the dual representation of the atomic SU(2) algebra (37) is provided by the following generators

$$\begin{aligned} J_z^{(\phi)} &= \sum_m m|\phi_m\rangle\langle\phi_m|, \\ J_+^{(\phi)} &= \sum_m \sqrt{2-m(m+1)}|\phi_{m+1}\rangle\langle\phi_m|, \\ J_-^{(\phi)} &= \sum_m \sqrt{2-m(m-1)}|\phi_{m-1}\rangle\langle\phi_m|. \end{aligned} \tag{1.48}$$

Similar results can be obtained for an arbitrary atomic multipole transition in much the same way as above. For example, in the case of the excited atomic state with $j = 2$, the representation of the $SU(2)$ algebra takes the form

$$\begin{aligned} J_+ &= \sqrt{2}|2\rangle\langle 1| + \sqrt{3}|1\rangle\langle 0| + \sqrt{3}|0\rangle\langle -1| + \sqrt{2}|-1\rangle\langle -2| \\ J_- &= \sqrt{2}|1\rangle\langle 2| + \sqrt{3}|0\rangle\langle 1| + \sqrt{3}|-1\rangle\langle 0| + \sqrt{2}|-2\rangle\langle -1| \\ J_z &= 2|2\rangle\langle 2| + |1\rangle\langle 1| - |-1\rangle\langle -1| - 2|-2\rangle\langle -2|, \end{aligned}$$

where $|n\rangle \equiv |m = n\rangle$. Corresponding exponential of the phase operator is

$$\varepsilon = |2\rangle\langle 1| + |1\rangle\langle 0| + |0\rangle\langle -1| + |-1\rangle\langle -2| + e^{i\psi}|-2\rangle\langle 2|.$$

In this case, the eigenvalues of the phase variable take the following five independent values

$$\phi_m = \frac{\psi + 2m\pi}{5}, \qquad m = 2, 1, \ldots, -2.$$

In the general case of an arbitrary integer $j \geq 1$, the number of independent eigenvalues of the phase variable ϕ_m is (2j + 1).

1.3.3 The $SU(2)$ atomic phase states, EPR paradox and entanglement

The above formalism of $SU(2)$ phase states can be used in a number of problems of quantum physics. As an illustrative example of great importance, consider the so-called Einstein-Podolsky-Rosen (EPR) paradox **[73]** (also see discussion in [14, 15, 74, **75]**). The EPR paradox touches the conceptual problems of reality and locality and existence of hidden variables in quantum physics as well as the more technological aspects of quantum cryptography **[34].**

In the original EPR gedanken experiment [73], a two-component system, consisting of two spin $1/2$ particles, is considered. Up to some time t_0, these particles are taken to be in a bounded state of zero angular momentum. At t_0, the binding is taken off without any disturbance of the spin states. Then, the separated particles move off in the opposite directions. Since the particles are in the common quantum state, the measurement of one chosen variable of particle one, moving "to the left", completely determines the outcome of a measurement of corresponding variable of particle two, moving "to the right".

Before making the measurement, the system is supposed to be in the EPR state which is also called the entangled state. It is described by the wave function of the form

$$|\Psi_{\pm}^{(EPR)}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_L\downarrow_R\rangle \pm |\downarrow_L\uparrow_R\rangle). \tag{1.49}$$

Here $|\updownarrow_{L,R}\rangle$ denotes the spin up or down state of the left or right particle. The two state vectors in the right-hand side of (49) form a basis of the corresponding Hilbert space in which we can define the representation of the $SU(2)$ algebra by the following generators

$$\begin{aligned} J_+ &= |\uparrow_L\downarrow_R\rangle\langle\downarrow_L\uparrow_R|, \\ J_- &= |\downarrow_L\uparrow_R\rangle\langle\uparrow_L\downarrow_R|, \\ J_z &= \frac{1}{2}(|\uparrow_L\downarrow_R\rangle\langle\uparrow_L\downarrow_R| - |\downarrow_L\uparrow_R\rangle\langle\downarrow_L\uparrow_R|), \end{aligned} \tag{1.50}$$

so that these operators obey the commutation relations (38) as well as the condition (39). Hence, the operators (47) admit a polar decomposition of the form discussed in previous Subsection. In particular, the exponential of the $SU(2)$ phase operator (41) takes the form

$$\varepsilon = |\uparrow_L\downarrow_R\rangle\langle\downarrow_L\uparrow_R| + e^{i\psi}|\downarrow_L\uparrow_R\rangle\langle\uparrow_L\downarrow_R|, \tag{1.51}$$

where ψ is again an arbitrary real reference phase. The $SU(2)$ phase states of the type of (45), (46) which are defined to be the eigenstates of the operator (51), have the form

$$|\phi_\sigma\rangle = \frac{1}{\sqrt{2}}(|\uparrow_L\downarrow_R\rangle + e^{i\phi_\sigma}|\downarrow_L\uparrow_R\rangle), \tag{1.52}$$

where

$$\phi_\sigma = \frac{\psi}{2} + \sigma\pi, \qquad \sigma = 0, 1.$$

It is now easily seen that, at $\psi = 0$, the $SU(2)$ phase states (52) coincide with the EPR states (49). Thus, the EPR states can be interpreted as the eigenstates of the exponential of the phase operator (51) of the $SU(2)$ algebra (50). The corresponding

quantum phase operator takes the form

$$\hat{\phi} = \frac{\psi + \pi}{2} \times \mathbf{1} - \frac{\pi}{2} e^{-i\psi/2} \varepsilon, \tag{1.53}$$

where

$$\mathbf{1} \equiv |\uparrow_L \downarrow_R\rangle\langle\uparrow_L \downarrow_R| + |\downarrow_L \uparrow_R\rangle\langle\downarrow_L \uparrow_R|.$$

By construction, the operator (53) describes the relative phase between the two EPR states (49).

There are many different physical realizations of the EPR or entangled states in optics and condensed matter physics. For example, the creation of two photons with different helicities by a single atom in the process of cascade decay of transition of the type $j \to j' = 0$ leads to the polarization-entangled state of the photons, leaving the atom in opposite directions (e.g., see Sec. 12.14.1 in [14]).

Another important example of entanglement is provided by the system of two two-level atoms in an optical resonator [76]. Such a system can be described by the following Jaynes-Cummings Hamiltonian

$$\begin{aligned} H &= H_0 + Hint, \\ H_0 &= \sum_{f=1,2} \omega_f |e_f\rangle\langle e_f| + \omega a^+ a, \\ H_{int} &= \sum_{f=1,2} i\gamma_f (|e_f\rangle\langle g_f| a - a^+ |g_f\rangle\langle e_f|). \end{aligned} \tag{1.54}$$

Here index f labels the atom in the cavity, $|e_f\rangle(|g_f\rangle)$ is the excited (ground) state of the corresponding atom, γ_f is the atom-field coupling constant, and the operators a^+ and a describe the cavity photons. Among the eigenstates of (54)

$$\begin{aligned} |\psi_0\rangle &= \frac{1}{\sqrt{\gamma_1^2 + \gamma_2^2}} (\gamma_1 |0; g_1; e_f\rangle - \gamma_2 |0; e_1; g_2\rangle), \\ |\psi_\pm\rangle &= \frac{1}{\sqrt{2}} \left(1; g_1; g_2\rangle \pm \frac{i}{\sqrt{\gamma_1^2 + \gamma_2^2}} (\gamma_2 |0; g_1; e_2\rangle + \gamma_1 |0; e_1; g_2\rangle \right), \end{aligned} \tag{1.55}$$

there is a maximally entangled atomic state $|\psi_0\rangle$ which, under the assumption that $\gamma_1 = \gamma_2$, takes the form

$$\begin{aligned} |\psi_0\rangle &= |0\rangle \otimes |\psi_{ent}\rangle, \\ |\psi_{ent}\rangle &= \frac{1}{\sqrt{2}} (|g_1; e_2\rangle - |e_1; g_2\rangle) \end{aligned} \tag{1.56}$$

similar to EPR state (49). In the above formulas we use the following notations:

$$|0; e_f; g_{f' \neq f}\rangle = |0\rangle_{field} \otimes |e_f\rangle \otimes |g_{f'}\rangle$$

$$|1; g_1; g_2\rangle = |1\rangle_{field} \otimes |g_1\rangle \otimes |g_2\rangle.$$

To establish contact with the $SU(2)$ phase states, we can consider the following representation of generators of the atomic $SU(2)$ algebra

$$\begin{aligned} J_+ &= |e_1\rangle\langle e_2|, \\ J_- &= |e_2\rangle\langle e_1|, \\ J_z &= \frac{1}{2}(|e_1\rangle\langle e_1| - |e_2\rangle\langle e_2|, \end{aligned}$$

similar to (50). Then, the exponential of the phase operator takes the form (for simplicity, we put here $\psi = 0$)

$$\varepsilon = |e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|.$$

It is now seen that the maximally entangled atomic state in (55) is again the SU(2) quantum phase state.

An interesting example of entanglement in condensed matter is represented by the formation of Cooper pairs in conventional superconductors. It is well known that the electron-phonon interaction in metals can lead to formation of collective quantum states of paired electrons with opposite spins and linear momenta [77]. In the simplest quasi-spin form, the system can be specified by the Hamiltonian [78, 79]

$$H = \sum_p E_p \sigma_p^z - \sum_{p,p'} J_{pp'} \sigma_p^- \sigma_{p'}^+, \tag{1.57}$$

where E_p denotes the energy spectrum depending on the momentum p of electrons, $J_{pp'}$ is the effective coupling constant, and the Pauli operators

$$\sigma_p^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_p^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_p^z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

correspond to the pairs of electrons with opposite spins and momenta. Since the Pauli operators obey the commutation relations

$$[\sigma_p^-, \sigma_{p'}^+] = 2\sigma_p^z \delta_{pp'}, \qquad [\sigma_p^z, \sigma_{p'}^{\mp}] = \pm\sigma_p^{\mp}\delta_{pp'}$$

which coincide with (38), by performing an analysis similar to that described in previous Subsection, we get

$$\varepsilon_p = \sigma_p^- + e^{i\psi}\sigma_p^+.$$

This is the $SU(2)$ exponential of the phase operator similar to (41) defined for each p.

It is also known that, in the so-called thermodynamic limit, when the number of electrons tends to infinity at constant density, the state of the system with the

quasi-spin BCS Hamiltonian (57) is the eigenstate of the trial Hamiltonian, in which the interaction part of (57) is changed by the operator [79]

$$H_{int} = -T_c \sum_p (\sigma_p^- \Delta + \sigma_p^+ \Delta^* - |\Delta|^2),$$

where T_c is some constant related to $J_{pp'}$ in (57) and **A** is the complex parameter, characterizing the gap in the spectrum of eigenenergy and depending on the temperature. Thus, at $\psi = -2$ arg A, the superconducting state (the eigenstate of ε_p) is the SU(2) quantum phase state, describing entangled electrons with opposite spins and momenta (Cooper pair). Therefore, the phase transition into the superconducting state can be interpreted as the creation of collective entangled state of electrons.

1.3.4 Resume

(i) For the electric dipole radiation described by the Jaynes-Cummings Hamiltonian (34), the polarization of photons at $kr \to 0$ is defined by the quantum number $m = 0$, fl, describing the excited atomic state.

(ii) For any atomic multipole transition, the excited state can be described in terms of the dual representation of corresponding SU(2) algebra, describing the azimuthal quantum phase of the angular momentum. In particular, the exponential of the phase operator and phase states can be constructed. The quantum phase variable has a discrete spectrum with (2j + 1) different eigenvalues.

(iii) In a special case of $j = 1/2$, the eigenstates of the exponential of the phase operator coincide with the EPR (entangled) states, which can be interpreted as the $SU(2)$ phase state.

1.4 QUANTUM PHASE OF MULTIPOLE RADIATION

No, my dear Watson, the two events are connected • must be connected. It is for us to find the connection.

-Sir Arthur Conan Doyl: The Second Stain

1.4.1 Conservation of angular momentum in the process of radiation

We now turn to the problem of the SU(2) quantum phase of multipole radiation. As a particular example of some considerable interest, we investigate the electric dipole field. All other types of the multipole radiation can be considered in the same way.

In Section 3.2, we introduced the atomic quantum phase states through the use of the representation of the $SU(2)$ algebra (37) and dual representation (48), corresponding to the angular momentum of the excited atomic state. The multipole

radiation emitted by atoms carries the angular momentum of the excited atomic state and can also be specified by the angular momentum [2, 26, 27]. The bare operators of the angular momentum of the electric dipole radiation have the form

$$\begin{aligned} M_+ &= \sqrt{2}(a_+^+ a_0 + a_0^+ a_-), \\ M_- &= \sqrt{2}(a_0^+ a_+ + a_-^+ a_0), \\ M_z &= \sum_{m=-1}^{1} m a_m^+ a_m. \end{aligned} \tag{1.58}$$

This result can be obtained by canonical quantization of the components of classical angular momentum (9) [2]. Hereafter in this Section we use the following notation:

$$a_m \equiv a_{Ek1m}.$$

Taking into account the commutation relations (23), it is easy to check that

$$[M_+, M_-] = 2M_z, \qquad [M_z, M_\pm] = \pm M_\pm, \tag{1.59}$$

so that the operators (58) form a representation of the $SU(2)$ sub-algebra in the Weyl-Heisenberg algebra (18) of the electric dipole photons.

The electric dipole photons, as well as the operators (58), are defined in the Hilbert space

$$\mathcal{H}_{field} = \bigotimes_{m=-1}^{1} \mathcal{H}_m, \tag{1.60}$$

where each sub-space $\mathcal{H}_m$ is spanned by the countable set of Fock vectors $|n_m\rangle$ ($n_m = 0, 1, 2, \ldots$), which obey the ortbogonality condition

$$\langle n'_{m'} | n_m \rangle = \delta_{nn'} \delta_{mm'}$$

and the completeness condition

$$\bigotimes_{m=-1}^{1} \sum_{n_m=0}^{\infty} |n_m\rangle\langle n_m| = \mathbf{1}. \tag{1.61}$$

Here 1 the unit operator, acting in (60). Unlike (39),

$$M^2 = M_z^2 + M_+ M_- - M_z \neq \mathbf{1}$$

in the whole Hilbert space (60). In other words, there is no isotype representation [52] of (58) in (60). Therefore, the polar decomposition of the $SU(2)$ sub-algebra (58) in the Weyl-Heisenberg algebra of electric dipole photons cannot be constructed in the way discussed in Section 3.2.

At the same time, we know that the photons carry information obtained from the atom in the process of generation. This information is transmitted through the conservation laws. In particular, the photon carries the angular momentum of the excited state because

$$[(J_\alpha + M_\alpha, H] = 0. \tag{1.62}$$

Here J_α denotes the atomic $SU(2)$ generators (37) with $\alpha = z, \pm$, M_α is the component of the field angular momentum operator (58), and H is the Jaynes-Cummings model Hamiltonian (34).

Since the atomic $SU(2)$ quantum phase, discussed in Section 3.2, is defined by the angular momentum of the excited atomic state, the conservation law (62) can be used to determine the field counterpart of the exponential of the phase operator (41) and other operators referred to the $SU(2)$ quantum phase [36, 46]. For example, it is easily seen that the operator

$$\varepsilon_{rad} = a_+^+ a_0 + a_0^+ a_- + a_-^+ a_+ \tag{1.63}$$

complements the atomic exponential of the phase operator (41) (at $\psi = 0$) with respect to the integral of motion:

$$[(\varepsilon_a + \varepsilon_{rad}), H] = 0. \tag{1.64}$$

The operator (63) can be considered as the result of "mapping" of the atomic exponential of the $SU(2)$ phase operator (41) on the field variables through the use of the integral of motion (64). Unlike (41), it is not unitary

$$\varepsilon_{rad}\varepsilon_{rad}^+ \neq 1,$$

but it is a normal operator

$$[\varepsilon_{rad}, \varepsilon_{rad}^+] = 0,$$

commuting with the total number of photons

$$[\varepsilon_{rad}, \sum_m a_m^+ a_m] = 0. \tag{1.65}$$

In the same way, it is easy to show that the operator constructions

$$\varepsilon_{rad} + \varepsilon_{rad}^+$$

and

$$-i(\varepsilon_{rad} - \varepsilon_{rad}^+)$$

complement the atomic cosine and sine of the $SU(2)$ phase operators (42) with respect to the integrals of motion with the atom-field Hamiltonian (34).

1.4.2 Dual representation of dipole photons

We now turn to the construction of the dual representation of the photon operators, providing the field counterpart of the $SU(2)$ phase representation of the atomic variables. It is easily seen that the atomic exponential of the $SU(2)$ phase operator (41) takes (in the representation of dual states (46)) the following diagonal form

$$\varepsilon_a^{(\phi)} = \sum_{m=-1}^{1} e^{i\phi_m} |\phi_m\rangle\langle\phi_m|, \tag{1.66}$$

where ϕ_m takes the values (46) (hereafter we put $\psi = 0$ without loss of generality). Thus, the dual representation of the atomic operators leads to the diagonal form of the exponential of the phase operator.

In turn, the field operator (63), representing the field counterpart of (41), can be diagonalized by the following Bogolubov type [80] canonical transformation [46]

$$\begin{aligned} \mathbf{a}, &= \frac{1}{\sqrt{3}} \sum_{m'=-1}^{1} e^{-im'\phi_m} \mathbf{a}_{m'}, \\ \mathbf{a}, &= \frac{1}{\sqrt{3}} \sum_{m'=-1}^{1} e^{im'\phi_m} a_{m'}, \end{aligned} \tag{1.67}$$

which has the form of finite Fourier transformation with ϕ_m defined in (46). It follows from the commutation relations (23) that

$$[\mathbf{a}_m, \mathbf{a}",,] = \delta_{mm'}. \tag{1.68}$$

Hence, the operators a in (67) also form a representation of the Weyl-Heisenberg algebra of the electric dipole photons. Employing this transformation (67) then gives the diagonal representation of the operator (63)

$$\varepsilon_{rad}^{(\phi)} = \sum_{m=-1}^{1} e^{i\phi_m} \mathbf{a}_m^+ \mathbf{a}_m \tag{1.69}$$

similar to (66). It is now a straightforward matter to arrive at the integral of motion

$$[(\varepsilon_a^{(\phi)} + \varepsilon_{rad}^{(\phi)}), H] = 0. \tag{1.70}$$

By construction, it corresponds to (64) in the dual representation of the dynamical variables for the atom and radiation field. This integral of motion reflects the fact that the $SU(2)$ phase information is also transmitted from the atom to photon in the

process of generation. In other words, the integral of motion (70) is responsible for the mapping of the atomic $SU(2)$ phase on the field variables. Therefore, one can choose to interpret a, and $\mathbf{a}_m^+$ in the canonical transformation (67) as the annihilation and creation operators of the electric dipole photons ***with given radiation phase*** [46].

As can be seen from tbe transformation (67), the operators a, obey the same stability condition (25) as a,:

$$\forall m,\ m' \qquad a_m|0\rangle = \mathbf{a}_{m'}|0\rangle = 0, \tag{1.71}$$

where the dipole vacuum state is defined as follows

$$|0\rangle \equiv \bigotimes_{m=-1}^{1} |0_m\rangle.$$

Hence, the creation operators $\mathbf{a}_m^+$ in (67) can be used to generate the Fock number states in the "phase representation"

$$|\nu_m\rangle = \frac{1}{\sqrt{\nu_m!}}(\mathbf{a}_m^+)^{\nu_m}|0\rangle, \tag{1.72}$$

such that

$$\mathbf{a}_m^+\mathbf{a}_m|\nu_m\rangle = \nu_m|\nu_m\rangle, \qquad \nu_m = 0, 1, \cdots$$

and

$$\langle\nu_m|\nu'_{m'}\rangle = \delta_{mm'}\delta_{\nu\nu'}, \qquad \bigotimes_{m=-1}^{1}\sum_{\nu_m}|\nu_m\rangle\langle\nu_m| = \mathbf{1}.$$

The unit operator here coincides with (61). Thus, the states (72) form a basis in the Hilbert space (60) dual to the basis of conventional number states $|n_m\rangle$. In analogy to the atomic phase states (46), we call (72) ***the radiation phase states*** of the electric dipole photons. It follows from (69) that the radiation phase states (72) are the eigenstates of the operator $\varepsilon_{rad}^{(\phi)}$:

$$\varepsilon_{rad}^{(\phi)}|\nu_m\rangle = \nu_m e^{i\phi_m}|\nu_m\rangle. \tag{1.73}$$

In contrast to the relation (45), the eigenvalues of $\varepsilon_{rad}^{(\phi)}$ in (73) contain, in addition to the exponential, a factor of ν_m, describing tbe number of photons in a given radiation phase state. Thus, this is an non-normalized exponential of the phase operator.

The above results lead to the conclusion that the radiation phase states (72) are dual to the conventional Fock number states $|n_m\rangle$. In turn, the operators (67) form the representation of the Weyl-Heisenberg algebra of the electric dipole photons dual to the operators a, and a_m^+ [46].

Although the canonical transformation (67) has the very simple form of the finite Fourier transformation, the connection between the conventional number states and the radiation phase states (72) is not simple:

$$|\nu_m\rangle = \sqrt{\frac{\nu_m!}{3^{\nu_m}}} \sum_{n_0=0}^{\nu_m} \sum_{n_+=0}^{\nu_m - n_0} \frac{\exp[(i(\nu_m - n_0 - 2n_+)\phi_m}{\sqrt{n_0! n_+! (\nu_m - n_0 - n_+)!}} \times |n_+; n_0; \nu_m \text{ - n o - n +)} . \tag{1.74}$$

It is interesting that the "dual" coherent states

$$|\alpha^{(a)}\rangle = \prod_m D_m^{(a)}(\alpha^{(a)})|0\rangle,$$
$$|\alpha^{(\mathbf{a})}\rangle = \prod_m D^{(\mathbf{a})}(\alpha^{(\mathbf{a})})|0\rangle$$

are equivalent up to the following transformation of the parameters:

$$\alpha_{m'}^{(a)} = \frac{1}{\sqrt{3}} \sum_m e^{-im'\phi_{m'}} \alpha_{m'}^{(\mathbf{a})}$$
$$\alpha_{m'}^{(\mathbf{a})} = \frac{1}{\sqrt{3}} \sum_m e^{im\phi_{m'}} \alpha_m^{(a)},$$

similar to the canonical transformation (67). Here

$$D_m^{(a)} = \exp(\alpha_m^{(a)} a_m^+ - H.c.)$$
$$D_m^{(\mathbf{a})} = \exp(\alpha_m^{(\mathbf{a})} \mathbf{a}_m^+ - H.c.)$$

are the "dual" Glauber displacement operators [81]. If we consider, as an example, the state $|\alpha_+; 0; 0)$ of the electric dipole radiation with only one component $m = +1$, we will see that it is represented by the dual coherent state

$$\bigotimes_m |\alpha_m^{(\mathbf{a})}\rangle, \qquad \alpha_m^{(\mathbf{a})} = \frac{1}{\sqrt{3}} \alpha_+^{(a)} e^{i2m\pi/3},$$

in which all the three "phase" components of the electric dipole radiation are in the coherent states.

The dual representation of the photon operators (67) reflects the transmission of "phase information" from the atomic transition to the radiation field via the integral of motion (70). This statement can be illustrated with the aid of the Jaynes-Cummings model (34). Employing the atomic phase states (46), we can introduce the dual representation of the atomic operators (35) as follows

$$R_{mm'}^{(\phi)} \equiv |\phi_m\rangle\langle\phi_{m'}|, \qquad R_{mg}^{(\phi)} \equiv |\phi_m\rangle\langle g|. \tag{1.75}$$

Then, the simultaneous use of the dual representation of the atomic operators (75) and the canonical transformation of photon operators (67) leads to the following form of the Jaynes-Cummings Hamiltonian (34):

$$H^{(\phi)} = H_0^{(\phi)} + H_{int}^{(\phi)},$$
$$H_0^{(\phi)} = \sum_{m=-1}^{1} [\omega \mathbf{a}_m^+ \mathbf{a}_m + \omega_0 R_{mm}^{(\phi)}],$$
$$H_{int}^{(\phi)} = +g(R_{mg}^{(\phi)} \mathbf{a}_m + \mathbf{a}_m^+ R_{gm}^{(\phi)})] \qquad (1.76)$$

which has exactly the same operator structure as (34) in the dual representation [46]. Since the dual atomic operators (75) describe the transition between the atomic phase states and ground state, and the operators a, and $\mathbf{a}_m^+$ determine the annihilation and creation of photons with given radiation phase, the interaction term in (76) ***describes the transmission of the quantum phase information from the atom to photons.***

We now note that the quantum phase in the Jaynes-Cummings model has been examined in a huge number of papers (see, for a review, Refs. [39, 68]). Most of them are based on the approach proposed in the pioneering paper by Dirac [1] and developed by a number of authors. Among the principal contributions to the field, Pegg-Bamett approach [45] should be mentioned as the most popular in recent years. The main idea of the approach consists in defining the quantum phase operator first in a finite s-dimensional subspace of the infinite-dimensional Hilbert space $\mathcal{H}_{field}$ with subsequent formal limit transition $s \to \infty$ which is taken only after the averages have been calculated. In contrast, we consider the extended space of states $\mathcal{H}_a \otimes \mathcal{H}_{field}$ in which the quantum phase of radiation is defined ***by mapping of corresponding atomic operatorsfrom $\mathcal{H}_a$ into the whole Hilbert space $\mathcal{H}_{field}$ (60), using the conservation of angular momentum.*** In view of the dual form of the Jaynes-Cummings Hamiltonian (76), it is possible to say that the radiation phase is expressed in terms of what ***can be generated*** by a given quantum source.

Our consideration so far have been applied to the electric dipole radiation. It is straightforward to find the general form of the canonical transformation (67) in the same way as above. In the case of an arbitrary monochromatic pure (λ, j)-pole radiation we get [46]

$$\mathbf{a}_{\lambda kjm} = \frac{1}{\sqrt{2j+1}} \sum_{m'=-j}^{j} e^{im'\phi_{jm}} a_{\lambda kjm'}, \qquad \phi_{jm} = \frac{2\pi m}{2j+1}. \qquad (1.77)$$

Here we again put $\psi = 0$.

1.4.3 Structure of radiation phase

We now examine the spectrum of radiation phase constructed in previous Subsection. Consider the state

$$|\phi^{(rad)}\rangle = \bigotimes_{m=-1}^{1} |\nu_m\rangle, \tag{1.78}$$

where $|\nu_m\rangle$ is the radiation phase state (72). It is clear that (78) is the eigenstate of the operator (73). Since the operator (73) commute with the total number of photons

$$N = \sum_{m=-1}^{1} a_m^+ a_m = \sum_{m=-1}^{1} \mathbf{a}_m^+ \mathbf{a}_m,$$

the eigenstates and eigenvalues of $\varepsilon_{rad}^{(\phi)}$ can be specified by the index

$$\mathrm{n} = \sum_{m=-1}^{1} \nu_m,$$

describing the total number of photons in a given state (72), (74) and by an additional index ℓ, describing a given distribution of n photons over the three independent phase components of the electric dipole radiation in (72), (74). The total number of possible different values of ℓ, corresponding to a given n, is clearly

$$\frac{1}{2}(n+2)(n+1).$$

Then

$$\varepsilon_{rad}^{(\phi)} |\phi_{n\ell}^{(rad)}\rangle = e^{i\varphi_{n\ell}} \kappa_{n\ell} |\phi_{n\ell}^{(rad)}\rangle, \tag{1.79}$$

where $|\phi_{n\ell}^{(rad)}\rangle$ denotes the state (78) at given n and ℓ. The modulus of the eigenstates in (79) is determined as

$$\begin{aligned} \kappa_{n\ell}^2 = \langle \phi_{n\ell}^{(rad)} | \varepsilon_{rad}^{(\phi)} (\varepsilon_{rad}^{(phi)})^+ | \phi_{n\ell}^{(rad)} \rangle &= \sum_{mm'} \nu_m \nu_m' e^{i(m-m')2\pi/3} \\ &= \sum_m \nu_m^2 - (\nu_+\nu_0 + \nu_0\nu_- + \nu_-\nu_+) \\ &= n^2 + 2(\nu_+^2 + \nu_-^2) - 3n(\nu_+ + \nu_-) + 3\nu_+\nu_-. \end{aligned} \tag{1.80}$$

In turn, for the "phase eigenvalues" $\varphi_{n\ell}$ in (79) we get [46]

$$\tan\phi = \frac{\sqrt{3}(\nu_+ - \nu_-)}{2\nu_0 - (\nu_+ + \nu_-)}. \tag{1.81}$$

Taking into account the physical meaning of the atomic operators (41) and (66) and the integrals of motion (64), we can consider the field operators (63) and (69) as the non-normalized exponential operators of the radiation phase which, by construction, is the $SU(2)$ phase of the multipole (electric dipole) radiation. By performing a similar analysis to that described in Subsection 3.2, we can define the cosine and sine operators of the radiation phase as follows [36]

$$C_{rad} = K(\varepsilon_{rad} + \varepsilon_{rad}^{+}), \qquad S_{rad} = -iK(\varepsilon_{rad} + \varepsilon_{rad}^{+}), \tag{1.82}$$

where K is the normalization coefficient determined from the natural condition

$$\langle C_{rad}^2 + S_{rad}^2 \rangle = 1, \tag{1.83}$$

where $\langle \cdot \cdot \cdot \rangle$ is the averaging over the states of the electric dipole radiation under consideration. It is clear that C_{rad} and S_{rad} are commuting Hermitian operators so that corresponding physical quantities can be measured at once. In the dual representation provided by the canonical transformation (67), the operators (82) take the diagonal form

$$\begin{aligned} C_{rad}^{(\phi)} &= K \sum_{m=-1}^{1} \mathbf{a}_m^{+} \mathbf{a}_m \cos\phi_m, \\ S_{rad}^{(\phi)} &= K \sum_{m=-1}^{1} \mathbf{a}_m^{+} \mathbf{a}_m \sin\phi_m, \end{aligned} \tag{1.84}$$

Therefore, averaging over the phase states (78) we get

$$\begin{aligned} \langle \phi_{n\ell}^{(rad)} | C_{rad}^{(\phi)} | \phi_{n\ell}^{(rad)} \rangle &= 2\kappa_{n\ell} \cos\varphi_{n\ell}, \\ \langle \phi_{n\ell}^{(rad)} | S_{rad}^{(\phi)} | \phi_{n\ell}^{(rad)} \rangle &= 2\kappa_{n\ell} \sin\varphi_{n\ell}. \end{aligned}$$

According to the condition (83), we obtain

$$K = \frac{1}{2\kappa_{n\ell}}$$

in this case. Hence, the radiation phase states (78) are the eigenstates of the operators (84) which, due to their structure, can be interpreted as the cosine and sine of the radiation phase operators. It is interesting that the eigenvalues of the radiation phase variable defined by (81) belong to the interval $(0, 2\pi)$ and form a discrete set for any finite number of photons n and

$$\ell = 1, 2, \cdot \cdot \cdot, (n+1)(n+2)/2.$$

The first few eigenvalues are shown in Table 1 and Fig. 5.

It is not difficult to see that the vacuum averages of the operators (84) have the form

$$\langle 0|C_{rad}^{(\phi)}|0\rangle = \langle 0|S_{rad}^{(\phi)}|0\rangle = 0,$$

while the vacuum variances

$$\langle 0|(\Delta C_{rad}^{(\phi)})^2|0\rangle = \langle 0|(\Delta S_{rad}^{(\phi)})^2|0\rangle = \frac{1}{2},$$

Hence, as all one can expect, the vacuum distribution of the radiation phase is uniform.

Consider now the electric dipole radiation with coherent components $m = \text{fl}$, while the component m = 0 is in the vacuum state:

$$|\alpha\rangle = |\alpha_+\rangle \otimes |0\rangle \otimes |\alpha_-\rangle \tag{1.85}$$

at $|\alpha_+| = |\alpha_-| \equiv |\alpha|$. In this case, the condition (83) gives

$$K = \frac{1}{\sqrt{4|\alpha|^2(2 + |\alpha|^2)}},$$

so that

$$\begin{aligned} \langle C_{rad}^{(\phi)}\rangle &= \frac{|\alpha|}{\sqrt{2 + |\alpha|^2}} \cos \Delta_{+-}, \\ \langle S_{rad}^{(\phi)}\rangle &= \frac{|\alpha|}{\sqrt{2 + |\alpha|^2}} \sin \Delta_{+-}, \end{aligned} \tag{1.86}$$

where

$$\Delta_{+-} \equiv \arg \alpha_+ - \arg \alpha_-.$$

One can see that at $|\alpha| \to \infty$ we get

$$\langle C_{rad}^{(\phi)}\rangle \to \cos \Delta_{+-}, \qquad \langle S_{rad}^{(\phi)}\rangle \to \sin \Delta_{+-},$$

so that in the classical limit of infinitely many coherent photons the operators (113) define the cosine and sine of the phase difference between the two components of the radiation field (85). In turn, for the variances we get

$$\langle 0|(\Delta C_{rad}^{(\phi)})^2|0\rangle = \langle 0|(\Delta S_{rad}^{(\phi)})^2|0\rangle = \frac{2 + \cos \Delta_{+-}}{2(2 + |\alpha|^2)}. \tag{1.87}$$

Here the right-hand side tends to zero when $|\alpha| \to \infty$. Hence, the radiation phase has the natural classical limit.

At first sight, the equation (87) gives wrong limit at $|\alpha| \to 0$:

$$\lim_{|\alpha|\to 0} \langle 0|(\Delta C_{rad}^{(\phi)})^2|0\rangle = \lim_{|\alpha|\to 0} \langle 0|(\Delta S_{rad}^{(\phi)})^2|0\rangle = \frac{2 + \cos\Delta_{+-}}{4}. \tag{1.88}$$

In fact, this is an illusory contradiction. Due to the degeneration of the vacuum state with respect to the phase difference, the limit transition $|\alpha| \to 0$ should imply the averaging over all possible $\mathbf{A} \in (0, 2\pi)$ which leads to the natural value of the vacuum variances.

Consider now the case when $|\alpha_+| \neq |\alpha_-|$ in the state (85). Then, instead of (86) and (87), we get

$$\begin{aligned} \langle C_{rad}^{(\phi)}\rangle &= \frac{\cos \mathbf{A},-}{\& + |\alpha_+|^{-2} + |\alpha_-|^{-2}} \\ \langle S_{rad}^{(\phi)}\rangle &= \frac{\sin\Delta_{+-}}{\sqrt{1 + |\alpha_+|^{-2} + |\alpha_-|^{-2}}} \end{aligned} \tag{1.89}$$

and

$$\begin{aligned} \langle (\Delta C_{rad}^{(\phi)})^2\rangle = \langle (\Delta S_{rad}^{(\phi)})^2\rangle \\ = \frac{|\alpha_+|^2 + |\alpha_-|^2 + |\alpha_+||\alpha_-|\cos\Delta_{+-}}{2(|\alpha_+|^2 + |\alpha_-|^2 + |\alpha_+|^2|\alpha_-|^2)} \end{aligned} \tag{1.90}$$

respectively. Examine now the averages (89) and variance (90) as a function of $|\alpha_+|$ at fixed $|\alpha_-|$ [54]. We obtain

$$\begin{cases} \langle C_{rad}^{(\phi)}\rangle \to 0 \\ \langle S_{rad}^{(\phi)}\rangle \to 0 \\ \langle 0|(\Delta C_{rad}^{(\phi)})^2|0\rangle \to 1/2 \\ \langle 0|(\Delta S_{rad}^{(\phi)})^2|0\rangle \to 1/2 \end{cases} \tag{1.91}$$

because the operators (84) define properties of the relative phase (phase difference between the components) which does not exist when only component m = -1 is emitted. It is seen that, under the condition

$$|\alpha_+||\alpha_-| < \cos\Delta_{+-} \leq 1, \tag{1.92}$$

which can be realized in the strong quantum case of very low intensities, the value of the variances (90) can exceed the vacuum limit of $1/2$. The maximum in (90) is achieved at

$$|\alpha_+| = |\alpha_-| \times \frac{\sqrt{|\alpha_-|^4 + (1 + |\alpha_-|^2)\cos\Delta_{+-}} - |\alpha_-|^2}{(1+ \quad |\alpha_-|^2)\cos\Delta_{+-}}.$$

The dependence of the variances (90) on $|\alpha_+|$ at fixed $|\alpha_-|$ is shown in Fig. 6. The qualitative explanation of this effect of strong increase of quantum fluctuations in the low-intensity limit is based on the consideration of the probability to have a given phase difference A+-. At $|\alpha_+ = 0$, there is a uniform probability distribution in the system, causing the limit relations (91). Creation of very few photons of the mode $m = +1$ leads to the formation of some domains with almost equal probabilities having phase differences Δ_{+-} and $\Delta_{\pm} + \pi$. So, it looks like a "phase bunching" (for the bunching of photons, see Refs. [14, 15]). Further increase of $|\alpha_+|$ leads to formation of a more or less sharp probability distribution which cannot reach the h-function shape because the variances (90) achieve the saturation described by the expression

$$\lim_{|\alpha_+|\to\infty} \langle C_{rad}^{(\phi)}) \rangle = \lim_{|\alpha_+|\to\infty} \langle S_{rad}^{(\phi)}) \rangle = \frac{1}{2(1 + |\alpha_-|^2)},$$

which coincides with the classical limit only if $|\alpha_-| \to \infty$ as well. This means, that the presence of one quantum component in the coherent state (85) leads to quantum phase fluctuations.

Using the standard representation of Glauber coherent states in terms of the number states of photons [14, 8 1]

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

and making use of the relation (74), we can represent the coherent state (85) in the following way

$$|\alpha_+; 0; \alpha_-\rangle = \prod_{m=-1}^{1} e^{-i|\alpha_m^{(a)}|^2/2} \sum_{\nu_m=0}^{\infty} \frac{(\alpha_m^{(a)})^{\nu_m}}{\sqrt{\nu_m}} |\nu_m\rangle,$$

where the parameters $\alpha_m^{(a)}$ have been defined at the end of Subsection 4.1. Assume that $|\alpha_+| = |\alpha_-| \equiv |\alpha|$. Then

$$\sum_m |\alpha_m^{(a)}|^2 = 2|\alpha|^2$$

and the probability to observe the radiation field in a given phase state (72), (74) is

$$|\langle \nu_+; \nu_0; \nu_- | \alpha_+; 0; \alpha_- \rangle|^2$$

$$= e^{-2|\alpha|^2} \left(\frac{2}{3}\right)^n \frac{|\alpha|^{2n}}{\nu_+! \nu_0! \nu_-!} \prod_{m=-1}^{1} [1 + \cos(\Delta_{+-} - 2m\pi/3)]^{\nu_m}, \qquad (1.93)$$

where $n = \sum_m \nu_m$. It is easily seen that this probability tends to zero when $|\alpha| \to 0$ or $|\alpha| \to \infty$. This means that the eigenvalues of the radiation phase are distributed

uniformly over the interval $(0, 2\pi)$ in the vacuum state as well as in the classical limit of high intensity coherent state. Between these two extrema, the probability (93) has a maximum, which can be considerably high. It is interesting that the position of the maximum is completely determined by the mean number of photons $|\alpha_{max}|^2$ = n, while the magnitude depends also on the phase difference A+- (see Fig. 7).

Let us now calculate the probability to have a given value of the radiation phase in the coherent state under consideration. Consider, for example, the eigenvalue of the radiation phase $\varphi = 2\pi/3$. Employing equations (80) and (81) then gives the following properties of the states corresponding to this radiation phase:

$$n - 3\nu_- = \kappa_n, \qquad \nu_+ = n - 2\nu_-, \qquad \nu_0 = \nu_-.$$

It is easy to see that the states obeying these conditions have the following structure:

$$|n_p - 2k_p; k_p; k_p\rangle \tag{1.94}$$

where p is an integer and for each $n_p = 3p, 3p-1, 3p-2$ the numbers k_p take the values $k_p = 0, 1, \ldots, p-1$. For example, at $n_p = 10$, we get $p = 4$ and $n_p = 3p-2$, while the states (94) are

$$|10; 0; 0)' \qquad |8; 1; 1\rangle, \qquad |6; 2; 2;\rangle, \qquad |4; 3; 3).$$

Consider first the states $|n_p; 0; 0)$ in (94). Then, the probability (93) takes the value

$$\mathcal{P}_{n_p} \equiv |\langle n_p; 0; 0|\alpha_+; 0; \alpha_-\rangle|^2$$
$$\times e^{-2|\alpha|^2} \left(\frac{2}{3}\right)^{n_p} \frac{|\alpha|^{2n_p}}{n_p!}[1 + \cos(\Delta_{+-} - 2\pi/3)]^{n_p}.$$

It is then clear that $\mathcal{P}_{n_p}$ reaches its maximum at A+- = $2\pi/3$. Then, the total probability to have the phase states $|n_p; 0; 0)$ is

$$\mathcal{P} = \sum_{n_p=1}^{\infty} \mathcal{P}_{n_p} = e^{-2|\alpha|^2/3} - e^{-2|\alpha|^2} \geq 0 \tag{1.95}$$

(see Fig. 8). It is clear that this function P gives the lower bond of the total probability to observe the radiation phase $\varphi = 2\pi/3$ in the coherent state under consideration. The contribution of the other states (94) can be calculated in the same way.

1.4.4 Radiation phase in Jaynes-Cummings model

To illustrate the exchange of the phase information between the atomic transition and multipole field, consider the electric dipole Jaynes-Cummings model (34). Assume that the field consists of two circularly polarized components in a coherent state each. The atom is supposed to be initially in the ground state. Then, the time-dependent

wave function of the system has the form [53]

$$|\Psi(t)\rangle = \sum_{n_+,n_-} P(n_+,n_-)[\cos(g\sqrt{nt})|0,\ 0\rangle + (\alpha_+|1,\ 1\rangle + \alpha_-|1,\ -1\rangle)\xi(n_+,n_-)]|n_+n_-\rangle, \quad (1.96)$$

where $|n_m\rangle$ denotes the Fock number state with n_m photons with the quantum number m, $|1,$ m) and $|0,$ 0) are the excited and ground atomic states respectively, g is the coupling constant in (34), n $= n_+n_-$,

$$P(n_+,n_-) = e^{-(|\alpha_+|^2-|\alpha_-|^2)/2}\frac{\alpha_+^{n_+}\alpha_-^{n_-}}{\sqrt{n_+!n_-!}},$$

and

$$\xi(n_+,n_-) = \frac{\sin(gt\sqrt{n+1})}{\sqrt{n+1}}e^{-i(n+1)\omega t}.$$

Then, the mean values of the operators (82), describing the cosine and sine of the radiation phase, take the form

$$\langle C_{rad}\rangle_t = \sqrt{\frac{|\alpha_+|^2|\alpha_-|^2}{|\alpha_+|^2|\alpha_-|^2 + |\alpha_+|^2|\alpha_-|^2\ |\alpha_+|^2 + |\alpha_-|^2}}\langle a_+^+a_+ + a_-^+a_-\rangle_t \cos\Delta_{+-},$$

$$\langle S_{rad}\rangle_t = \sqrt{\frac{|\alpha_+|^2|\alpha_-|^2}{|\alpha_+|^2|\alpha_-|^2 + |\alpha_+|^2|\alpha_-|^2\ |\alpha_+|^2| + \alpha_-|^2}}\langle a_+^+a_+ + a_-^+a_-\rangle_t \sin\ \Delta_{+-}. \quad (1.97)$$

Here

$$\langle a_+^+a_+ + a_-^+a_-\rangle_t = |\alpha_+|^2 + |\alpha_-|^2 - \sum_{n_+,n_-=0}^{\infty}|P(n_+,n_-)|^2\sin^2 gt\sqrt{n_+ + n_-}$$

and $\Delta_{+-} \equiv \arg\alpha_+ -$ arg CL. Thus, the averages (97) describe the Rabi oscillations of cosine and sine of the phase difference between the two coherent components of the field. Corresponding variances have tbe form

$$\langle(\Delta C_{rad})^2\rangle_t = \frac{\langle a_+^+a_+ + a_-^+a_-\rangle_t}{|\alpha_+|^2 + |\alpha_-|^2}[\langle(\Delta C_{rad})^2\rangle_0 + \beta_C Q],$$

$$\langle(\Delta S_{rad})^2\rangle_t = \frac{\langle a_+^+a_+ + a_-^+a_-\rangle_t}{|\alpha_+|^2 + |\alpha_-|^2}[\langle(\Delta S_{rad})^2\rangle_0 + \beta_S Q]. \quad (1.98)$$

Here

$$\langle(\Delta C_{rad})^2\rangle_0 = -\frac{1}{2}\cos\ A + \text{-},$$

$$\langle(\Delta S_{rad})^2\rangle_0 = \frac{|\alpha|^2 + |\alpha_-|^2 - |\alpha_+||\alpha_-|}{|\alpha|^2 + |\alpha_-|^2 + |\alpha_+||\alpha_-|}\cos\Delta_{+-}$$

and

$$\beta_C = \frac{|\alpha_+||\alpha_-|}{(|\alpha|^2 + |\alpha_-|^2 + |\alpha_+||\alpha_-|)(|\alpha|^2 + |\alpha_-|^2 + |\alpha_+||\alpha_-|)}\cos^2\Delta_{+-},$$

$$\beta_C = \frac{|\alpha_+||\alpha_-|}{(|\alpha|^2 + |a\text{-}12 + |\alpha_+||\alpha_-|)(|\alpha|^2 + |\alpha_-|^2 + |\alpha_+||\alpha_-|)}\sin^2\Delta_{+-}$$

In (98), Q is the Mandel's factor [14], describing the deviation of the photon statistics from the Poisson distribution for the total intensity:

$$Q = \frac{\langle[\Delta(a_+^+a_+ + a_-^+a_-)]^2\rangle - \langle(a_+^+a_+ + a_-^+a_-)\rangle}{\langle(a_+^+a_+ + a_-^+a_-)\rangle}$$

It is seen that the time-averaged Mandel's Q-factor is always positive, which shows the super-Poisson number distribution for the total field.

Since $\langle C_{rad}\rangle_t$ and $\langle(\Delta C_{rad})^2\rangle_t$ can betransformedinto $\langle S_{rad}\rangle_t$ and $\langle(\Delta S_{rad})^2\rangle_t$, respectively, by the change of phase difference

$$\Delta_{+-} \to \Delta_{+-} + \frac{k\pi}{2},$$

it is enough to examine only one pair of these functions. In Fig. 9, the Rabi oscillations of the variance ($(\Delta C_{rad})^2\rangle_t$ are shown as a function of time for different A , - . At small A+-, the collapse and revival picture of Rabi oscillations behaves quite typically for the Jaynes-Cummings model (e.g., see [68]), while the increase of Δ_{+-} leads to a confluence of the nearest revivals. The Rabi oscillations of $\langle C_{rad}\rangle_t$ have similar behavior.

It should be emphasized that the system under consideration completely describes the process of transmission of the phase information between the field and atom. Initially the atom is in the ground state with the angular momentum 0 and has no $SU(2)$ atomic phase at all. Absorption of photons induces an atomic phase which coincides with the phase difference between the two coherent components of the field. It can be concluded from a direct calculation of the expectation value of the atomic cosine operator (43) over the state (96):

$$\langle C_a\rangle_t = \sum_{n_+,n_-=0}^{\infty} |P(n_+,n_-)|^2 \frac{\sin^2(gt\sqrt{n+1})}{n+1}\cos\Delta_{+-}.$$

This again clearly demonstrates the one-to-one correspondence between the atomic SU(2) phase and the radiation phase.

Let us also stress that, according to (98), the variances of the cosine and sine of the radiation phase can be measured in the same way as the Mandel's Q-factor.

1.4.5 Radiation phase and Pegg-Barnett quantum phase

Following [54], let us compare the quantum statistical behavior of the radiation phase, constructed in the previous Subsections of this Section and that obtained within the Pegg-Barnett approach [45], which has received a lot of attention during the last ten years and has led to many important results (see, for a recent review Refs. [39, 40]). We use here the form of the Pegg-Bamett approach considered in Ref. [82]. The point is that Ref. [82] deals with a generalization of the Pegg-Barnett approach to the case of two circularly polarized modes. Then, the phase distribution over the phases of two circularly polarized modes is determined as follows,

$$P_\psi = |\langle \phi_+, \phi_- | \psi \rangle|^2,$$

where $|\phi_+, \phi_-\rangle$ is the Susskind-Glogower phase state [41] and $|\psi\rangle$ is the state of the radiation field. To establish the connection with the results already obtained in this Section, suppose that

$$|\psi\rangle = |\alpha_+\rangle \otimes |0_0\rangle \otimes |\alpha_-\rangle. \tag{1.99}$$

Since the formalism of the radiation phase is focused on the phase difference between the components, we need to use the distribution function for the relative phase

$$\phi = \phi_+ - \phi_-.$$

We use here the notations of Refs. [54, 82]. Referring to the procedure suggested in Refs. [82, 83] to cast the range of ϕ into 2π range from 4π range, we take

$$P_{2\pi} = \sum_{n=0}^{\infty} |\langle \phi^{(n)} | \psi \rangle|^2,$$

where

$$|\phi^{(n)}\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n_+=0}^{n} e^{i\phi n_+} |n_+\rangle \otimes |0_0\rangle \otimes |n - n_+\rangle.$$

Using this distribution function, one can calculate the mean value of any function $F(\phi)$ of the relative phase as follows [82]

$$\langle F(\phi) \rangle = \int_{-\pi}^{\pi} d\phi P_{2\pi}(\phi) F(\phi).$$

In the case under consideration, the relative phase represents the phase difference between the two circularly polarized components in coherent state. Then, for the

Pegg-Bamett cosine of the relative phase we get

$$\langle \cos \phi_{PB} \rangle = e^{(|\alpha_+|^2+|\alpha_-|^2)} \sum_{n_\pm=0}^{\infty} \frac{|\alpha_+|^{2n_+}|\alpha_-|^{2n_-}}{n_+!n_-!} \times \frac{Re(\alpha_+\alpha_-^*)}{\sqrt{(n_+ + 1)(n_- + 1)}}. \quad (1.100)$$

In turn, the average of the squared cosine takes the form

$$\langle \cos^2 \phi_{PB} \rangle = \frac{1}{2} + \frac{e^{(|\alpha_+|^2+|\alpha_-|^2)}}{2} \sum_{n_\pm=0}^{\infty} \frac{|\alpha_+|^{2n_+}|\alpha_-|^{2n_-}}{n_+!n_-!} \times \frac{Re(\alpha_+\alpha_-^*)}{\sqrt{(n_+ + 2)(n_+ + 1)(n_- + 2)(n_- + 1)}}. \quad (1.101)$$

Here ϕ_{PB} denotes the Pegg-Bamett quantum phase operator [45]. These expressions can be now compared with the results (89) and (90) for the radiation cosine and its variance. To clarify the difference between the two approaches, we represent (89) and (90) in the same manner as (100) and (101) [54]:

$$\langle C_{rad}^{(\phi)} \rangle = e^{(|\alpha_+|^2+|\alpha_-|^2)} \sum_{n_\pm=0}^{\infty} \frac{|\alpha_+|^{2n_+}|\alpha_-|^{2n_-}}{n_+!n_-!} \times \frac{Re(\alpha_+\alpha_-^*)}{\sqrt{|\alpha_+|^2 + |\alpha_-|^2 + |\alpha_+|^2|\alpha_-|^2}} \quad (1.102)$$

and

$$\langle (C_{rad}^{(\phi)})^2 \rangle = \frac{1}{2} + \frac{\langle C_{rad}^{(phi)} \rangle}{2\sqrt{|\alpha_+|^2 + |\alpha_-|^2 + |\alpha_+|^2|\alpha_-|^2}} + \frac{[Re(\alpha_+\alpha_-^*)]^2}{2(|\alpha_+|^2 + |\alpha_-|^2 + |\alpha_+|^2|\alpha_-|^2)} \sum_{n_\pm+0}^{\infty} \frac{|\alpha_+|^{2n_+}|\alpha_-|^{2n_-}}{n_+!n_-!}. \quad (1.103)$$

One can easily see that each term in the sums in equations (100) and (101) has different normalization, while in equations (102) and (103) all the terms have the same normalization factor related to our choice of the constant K in (82), (83). In addition, the equation (103) contains an extra term proportional to $\langle C_{rad}^{(\phi)} \rangle$. This term comes from the vacuum fluctuations related to the mode m = 0. This causes a striking difference when one of the modes m = ± 1 is in the quantum domain. Exactly, the existence of the "phase bunching" discussed in Section 4.3 is caused just by this term. At the same time, both approaches show the saturation of the variances when one of the intensities tends to infinity while the second if kept constant (Fig. 10). It is seen that the quantum fluctuations of the phase difference between the two

circularly polarized coherent fields with different helicity calculated with the aid of the radiation phase always exceed those calculated within the Pegg-Bamett approach.

This fact can be explained in the following way. The cutoff of the Hilbert space, which is a distinctive feature of the Pegg-Barnett approach [45], leads to an effective change of the algebraic properties of the photon operators. In fact, such a cutoff of the Fock basis leads to the definition of the unit operator which can be considered as some approximation of the Casimir operator of the $SU(2)$ sub-algebra, describing the angular momentum of radiation, but only in a particular sub-space of the Hilbert space of photons. Existence of the unit operator makes it possible to perform direct polar decomposition and determine the corresponding quantum phase properties. It can be clearly traced in Ref. [84]. At the same time, the cutoff procedure reduces the algebraic properties of photons responsible for the quantum fluctuations. The limit taken in the Pegg-Barnett approach after the calculation of all expectation values cannot completely restore these properties, which are especially important in the quantum domain.

Unlike the Pegg-Barnett approach, the definition of the radiation phase is based on the conservation laws for electromagnetic radiation and canonical transformation (67), which does not disturb the Weyl-Heisenberg algebra of the photon operators.

1.4.6 Radiation phase and Mandel's operational approach

We now note that the operators (69) and (84) introducing the radiation phase, are defined in terms of bilinear forms in the photon operators. At first sight, such a definition runs counter to the original idea by Dirac to determine the Hermitian quantum phase via ***linear*** forms in the photon operators [1] (also see [38, 42, 44]). Leaving aside Dirac's problem of existence of a Hermitian quantum phase variable of a harmonic oscillator, we should emphasize that the use of bilinear forms seems to be quite reasonable from the physical point of view. It can be argued in the following way.

First, the phase information is transmitted from the quantum source (atom) to photons via the conservation laws. In fact, there are only three physical quantities which are conserved in the process of radiation: energy, linear momentum, and angular momentum [26]. All of them are represented by the bilinear forms in the photon operators.

Second, the detection process is also based on the transmission of energy, linear momentum, and/or angular momentum from the photons to a detecting device [14]. In other words, the Hermitian bilinear forms in (84) corresponds to what can be emitted by the source and detected by a photodetector.

Let us stress that the operational definition of the quantum phase of radiation [47] is also based on the use of bilinear forms in the photon operators. In the simplest form, the idea of the operational approach to the phase difference can be illustrated with the aid of the two-port interferometer shown in Fig. 11 (see Refs. [14, 47] for more detailed discussion). The two incident monochromatic (or quasi-monochromatic) light beams are combined by a symmetric beam splitter oriented at 45^o to each

beam. The resultant intensities emerging from each output port are measured by the two photodetectors connected with a "comparator" (computer) like in the **Hanbury**-Brown-Twiss interferometer **[85]** (also see Refs. **[14,** 15, **86]).** Following **[47],** denote by a_1 and a_2 the photon annihilation operators, describing the field at the two input ports, and by as and a_4 the corresponding operators at the two output ports. Then

$$\begin{aligned} a_3 &= t a_1 + r' a_2, \\ a_4 &= r a_1 + t' a_2, \end{aligned}$$

where t and r denote the complex-amplitude transmittance and reflectance from one side of the beam splitter and r' and t' from the other side. The number of photons in the output modes is defined as follows

$$\begin{aligned} \hat{n}_3 &= |t|^2 \hat{n}_1 + |r|^2 \hat{n}_2 + t^* r' a_1^+ a_2 + t r'^* a_2^+ a_1, \\ \hat{n}_4 &= |r|^2 \hat{n}_1 + |t|^2 \hat{n}_2 + t' r^* a_1^+ a_2 + r t'^* a_2^+ a_1, \end{aligned}$$

where $\hat{n}_\ell \equiv a_\ell^+ a_\ell$. It is clear that

$$\hat{n}_4 - \hat{n}_3 = \exp[i \arg r - \arg r')] a_2^+ a_1 + \exp[i(\arg t' - \arg t)] a_1^+ a_2.$$

Hence, the sine and cosine of the phase difference between the two output beams *can* be defined as follows **[47]**

$$\begin{aligned} S_M &= K_1 (\exp[i(\arg r - \arg t')] a_2^+ a_1 + \exp[i(\arg t' - \arg r)] a_1^+ a_2), \\ C_M &= i K_2 (\exp[i(\arg r - \arg t')] a_2^+ a_1 - \exp[i(\arg t' - \arg r)] a_1^+ a_2), \end{aligned}$$

where K_1 and K_2 are some constants. It is clear that, apart from the exponential factors caused by the measuring device, the above equations for S_M and C_M have the operator structure similar to the cosine and sine operators of the radiation phase (82). Of course, the above discussed operational definition by **Mandel** et al **[47]** does not take into consideration the third component of the multipole radiation. Therefore the operators (104) do not commute with each other.

By virtue of the above discussion, one can conclude that the method of radiation phase , defining the quantum phase variable in terms of what can be emitted by the source, complements the operational phase which deals with what can be measured in a real experiment.

1.4.7 Phase properties of radiation in Fabry-Pérot resonator

Our consideration so far have applied to photons in an ideal spherical cavity. Consider now the very important case of interaction between a single atom with electric dipole transition and cavity field in the case of Fabry-Perot resonator formed by two parallel ideal reflecting mirrors. In this case, the cavity field can consist only of the photons propagating along the axis of resonator (z-axis) because all other photons should

leave the space limited by the mirrors. This means that the cavity photons have well defined direction and therefore they are in a state with given linear momentum (21), (22). Hence, the radiation emitted by the electric dipole transition consists of the two modes with m $= \pm 1$, while the radiation of the third mode m $= 0$ is forbidden. In this case, the photons with given helicity can be represented in terms of linearly polarized photons as follows [27]

$$a_\mu = \mp \frac{a_x \pm i a_y}{\sqrt{2}}, \qquad \mu = \pm 1. \tag{1.104}$$

Since the atom-field interaction in the Fabry-Pérot resonator is allowed for the two electric dipole transitions

$$|1, \pm 1\rangle \rightarrow |0, 0\rangle,$$

the interaction term in the electric dipole Jaynes-Cummings Hamiltonian (34) takes the form

$$H_{int} = g \sum_{\mu=\pm 1} (R_{\mu g} a_\mu + a_\mu^+ R_{g\mu}). \tag{1.105}$$

Then, the representation of the $SU(2)$ algebra corresponding to the excited state in (105) takes the form

$$\begin{aligned} J_z &= (R_{++} - R_{+-})/2, \\ J_+ &= R_{+-}, \\ J_- &= R_{-+} \end{aligned} \tag{1.106}$$

similar to (50). Then, the exponential of the atomic $SU(2)$ phase operator takes the form

$$\varepsilon = R_{+-} + e^{i\psi} R_{-+}, \tag{1.107}$$

such that

$$\varepsilon \varepsilon^+ = R_{++} + R_{--} = \mathbf{1}$$

and

$$\varepsilon^2 = e^{i\psi} \mathbf{1},$$

Here again ψ denotes an arbitrary reference phase. The atomic cosine and sine of the phase operators (42) are defined in the case under consideration as follows

$$C = (R_{+-} e^{-i\psi/2} + R_{-+} e^{i\psi/2}) \cos \frac{\psi}{2} = \varepsilon e^{-i\psi/2} \cos \frac{\psi}{2}$$

$$S = (R_{+-}e^{-i\psi/2} + R_{-+}e^{i\psi/2}) \sin\frac{\psi}{2} = \varepsilon e^{-i\psi/2} \sin\frac{\psi}{2}. \quad (1.108)$$

It is clear that these are commuting Hermitian operators.

In analogy to (45) and (46), we can define the atomic phase states as follows:

$$|\phi_\mu\rangle = \frac{1}{\sqrt{2}}(|1, +1\rangle + e^{i\phi_\mu}|1, -1\rangle), \quad (1.109)$$

where

$$\phi_\mu = \frac{\psi + (1-\mu)\pi}{2}, \qquad \mu = \pm 1. \quad (1.110)$$

It is seen that, apart from a factor of $e^{i\psi/2}$, the states (110) formally coincide with the EPR states (49). It is clear that the operator (107) is represented in terms of states (109) in the diagonal form

$$\varepsilon = \sum_{\mu=\pm 1} e^{i\phi_\mu}|\phi_\mu\rangle\langle\phi_\mu| \quad (1.111)$$

(cf. (66)).

Following the approach discussed in Section 4.1, we note that the field counterpart of (106) is provided by the operators

$$\begin{aligned} M_z &= (a_+^+ a_+ - a_-^+ a_-)/2, \\ M_+ &= a_+^+ a_-, \\ M_- &= a_-^+ a_+, \end{aligned} \quad (1.112)$$

which form the representation of the $SU(2)$ sub-algebra in the Weyl-Heisenberg algebra of photons. As well as (58), this is not an isotype representation. In analogy to (63), we introduce the field counterpart of (107)

$$\varepsilon_{rad} = a_+^+ a_- + e^{i\psi} a_-^+ a_+. \quad (1.113)$$

It is easy to verify that (113) complements the atomic operator (107) with respect to the integral of motion with the Jaynes-Cummings Hamiltonian (34) with the interaction term of the form (105). In analogy to (82), we introduce the cosine and sine of the radiation phase operators:

$$\begin{aligned} C_{rad} &= K(\varepsilon_{rad}^+ + \varepsilon_{rad}) = 2Ke^{-i\psi/2}\varepsilon_{rad} \cos\frac{\psi}{2}, \\ S_{rad} &= iK(\varepsilon_{rad}^+ - \varepsilon_{rad}) = 2Ke^{-i\psi/2}\varepsilon_{rad} \sin\frac{\psi}{2}, \end{aligned} \quad (1.114)$$

such that

$$[C_{rad}, S_{rad}] = 0,$$

and

$$[(C + C_{rad}), H] = [(S + S_{rad}), H] = 0.$$

Then, the condition (83) takes the form

$$\langle C_{rad}^2 + S_{rad}^2 \rangle = 4K^2 e^{-i\psi} \langle \varepsilon_{rad}^2 \rangle = 1. \tag{1.115}$$

In analogy to (66), the operator (113) is diagonalized by the canonical transformation of the form

$$\begin{aligned} \mathbf{a}_\mu &= \frac{1}{\sqrt{2}} \sum_{\mu'=\pm 1} e^{-i(1-\mu')\phi_\mu/2} \mathbf{a}_{\mu'}, \\ \mathbf{a}_\mu &= \frac{1}{\sqrt{2}} \sum_{\mu'=\pm 1} e^{i(1-\mu')\phi_\mu/2} a_\mu, \end{aligned} \tag{1.116}$$

where ϕ_μ is the phase variable (110).

Taking into account the equation (104), expressing the photons with given helicity in terms of linearly polarized photons, it is easily seen from (116) that we get

$$\mathrm{a}_+ = -a_x E^{i\psi/2}, \qquad \mathbf{a}_- = -ia_y e^{-i\psi/2}.$$

Hence, the dual representation of the photon operators (104) with given helicity coincides, apart from certain unimportant factors, with the photon operators with given linear polarization. In the representation (116), the operators (113) and (114) take the diagonal form

$$\begin{aligned} \varepsilon_{rad}^{(\phi)} &= \sum_{\mu=\pm 1} e^{i\phi_\mu} \mathbf{a}_\mu^+ \mathbf{a}_\mu \cos\frac{\psi}{2}, \\ C_{rad}^{(\phi)} &= 2K \sum_{\mu=\pm 1} \mathbf{a}_\mu^+ \mathbf{a}_\mu \cos\phi_\mu \cos\frac{\psi}{2}, \\ S_{rad}^{(\phi)} &= 2K \sum_{\mu=\pm 1} \mathbf{a}_\mu^+ \mathbf{a}_\mu \sin\phi_\mu \cos\frac{\psi}{2}, \end{aligned} \tag{1.117}$$

where the phase variable ϕ_μ is defined in (110). Since the condition (115) can now be done in a straightforward manner to yield

$$K = \frac{1}{2\cos(\psi/2)\sqrt{\langle(\mathbf{a}_+^+\mathbf{a}_+ - \mathbf{a}_-^+\mathbf{a}_-)^2\rangle}}, \tag{1.118}$$

the expressions for cosine and sine operators in (117) take the form

$$C_{rad}^{(\phi)} = \frac{\sum_\mu \mathbf{a}_\mu^+ \mathbf{a}_\mu \cos\phi_\mu}{\sqrt{\langle(\sum_\mu \mu \mathbf{a}_\mu^+ \mathbf{a}_\mu)^2\rangle}},$$
$$S_{rad}^{(\phi)} = \frac{\sum_\mu \mathbf{a}_\mu^+ \mathbf{a}_\mu \sin\phi_\mu}{\sqrt{\langle(\sum_\mu \mu \mathbf{a}_\mu^+ \mathbf{a}_\mu)^2\rangle}}. \quad (1.119)$$

This representation explicitly manifests the structure of the cosine and sine operators in terms of cosine and sine of phase variable (110).

In analogy to (72), (74) we now construct the dual Fock number states

$$|\nu_\mu\rangle = \frac{(\mathbf{a}_\mu^+)^{\nu_\mu}}{\sqrt{\nu_\mu!}}|0\rangle$$
$$= \sum_{k=0}^{\nu_\mu} \sqrt{\frac{\nu_\mu!}{2^{\nu_\mu} k!(\nu_\mu - k)!}} (-1)^{\nu_\mu - k} |k\rangle_+ \otimes |\nu_\mu - k\rangle_-. \quad (1.120)$$

Here ν_μ is an integer such that

$$\mathbf{a}_\mu^+ \mathbf{a}_\mu |\nu_\mu\rangle = \nu_\mu |\nu_\mu\rangle$$

and the states in the right-hand side correspond to the photons with given helicity. It is easy to show that the states (120) obey the orthonormality and completeness conditions.

Following the method described in Section 4.3, we introduce the state

$$|\phi^{(rad)}\rangle = \bigotimes_{\mu=\pm 1} |\nu_\mu\rangle \quad (1.121)$$

(see (78)), which can be interpreted as the radiation phase state in the resonator under consideration. For a given total number of photons $n = \sum_\mu \nu_\mu$, there are $n + 1$ different phase states with degenerated "eigenphase"

$$\varphi_{n\ell} = \frac{\psi}{2} + 2\ell\pi, \qquad \ell = 1, \cdots, n+1.$$

This result is obtained by analogy with (79)-(81). Thus, unlike the case of spherical cavity, the spectrum of the SU(2) phase of photons in the Fabry-Perot resonator is trivial.

To complete the comparison with the previous results obtained in Sections 4.3 and 4.5, let us average the cosine operator in (114) over the states with two circularly

polarized coherent modes $|\alpha_+, \alpha_-\rangle$. We get

$$\langle C_{rad}\rangle = \frac{2|\alpha_+||\alpha_-|\cos(\Delta_{+-}+\psi/2)}{|\alpha_+|^2+|\alpha_-|^2+4|\alpha_+|^2|\alpha_-|^2\cos^2(\Delta_{+-}+\psi/2)}\cos\frac{\psi}{2}. \quad (1.122)$$

It is clear that

$$\langle C_{rad}\rangle \to 0,$$

in the "vacuum limit" $|\alpha_\pm| \to 0$ independent of ψ, and

$$\langle C_{rad}\rangle \to \cos\frac{\psi}{2},$$

in the classical limit $|\alpha_\pm| \to \infty$. The behaviour of corresponding variance is shown in Fig. 12 and Fig. 13 in the cases of (a+ $| = |\alpha_-| \equiv |\alpha|$ and fixed $|\alpha_-|$ respectively. It is seen, that the variance shows the "normal" behaviour and that the "phase bunching", discussed in Section 4.3, does not exist in the case under consideration. In contrast to (90), the presence of the component $\mu = -1$ in the quantum state ($|\alpha_-|^2 \ll 1$) leads to the decrease of fluctuations.

1.4.8 Resume

(i) There is no isotype representation of the $SU(2)$ sub-algebra, describing the angular momentum of radiation, in the Weyl-Heisenberg algebra of photons. Therefore, the $SU(2)$ quantum phase of angular momentum of radiation field cannot be constructed in the same way as that of aquantummechanical system with finite number of degrees of freedom.

(ii) At the same time, the conservation of angular momentum in the process of radiation makes it possible to map the atomic phase variable into the field variables via corresponding integrals of motion.

(iii) The Weyl-Heisenberg algebra of multipole photons allows the dual representation in which we deal with the photons with given radiation phase (the $SU(2)$ phase of angular momentum) instead of standard photons with given projection of the angular momentum.

(iv) Through the use of the dual representations of the atomic and field operators, it is possible to construct an equivalent form of the Jaynes-Cummings Hamiltonian, describing the exchange of the "phase information" between the atom and radiation field.

(v) The radiation phase of multipole photons has discrete spectrum in the interval $(0, 2\pi)$. In the classical limit of high-intensity coherent field, the eigenvalues of the radiation phase are distributed uniformly over $(0, 2\pi)$.

(vi) The quantum fluctuations of the radiation phase manifest qualitativedifference from those calculated within the Pegg-Barnett approach. In particular, the "phase bunching" effect can be observed for a multipole radiation in a spherical cavity in

the quantum domain of low intensity. This effect does not occur in a linear cavity (Fahry-Perot resonator).

(vii) Our approach, leading to the definition of the radiation phase, is the natural complement of the Mandel's operational approach. The radiation phase is defined in terms of what can be transmitted from a quantum mechanical system to photon and vice versa.

1.5 POLARIZATION PROPERTIES OF MULTIPOLE RADIATION

It is not really difficult to construct a series of inferences, each depends upon its predecessor and each simple in itself.

-Sir Arthur Conan Doyl: The Dancing Men

1.5.1 Polarization of classical field

In previous Sections, we considered the polarization as a formal property of either plane or spherical waves of photons described by the corresponding index in the expansion (13) and (17). In this Section, we examine the quantum properties of polarization in more details. In particular, we show that the radiation phase of electric dipole radiation formally coincides with the inherent quantum phase of polarization which is the $SU(2)$ phase of spin of photons.

It is well known that the polarization measurements play an important role in optics and spectroscopy (e.g., see [87]). The description usually given of the polarization is a classical one, defining the polarization as a measure of *transversal anisotrophy* of the plane electromagnetic waves [57]. It is based on the fact that the field strengths (11) have only two symmetric spatial components. At the same time, these complex components may have different magnitude and phases. The quantitative description of polarization is provided by the so-called polarization matrix with the elements [14, 57]

$$P_{\sigma\sigma'} = E^*_\sigma(\vec{r})E_{\sigma'}(\vec{r}). \tag{1.123}$$

Here $E_\sigma(\vec{r})$ denotes the component of the positive-frequency part of the classical electric field strength. For monochromatic plane waves, in view of (13) we get

$$P_{\sigma\sigma'} = \gamma^2 a^*_{k\sigma} a_{k\sigma'}. \tag{1.124}$$

By definition [14, 25, 57], the diagonal elements in (26) and (27) give the contribution of the corresponding spatial components of the radiation field into the energy density, while the off-diagonal elements give the "phase" information concerning the phase difference between the components.

The polarization matrix (124) can also be represented in the helicity basis (16) at $\vec{\chi}_0 = \vec{k}/k$ as follows

$$P_{\mu\mu'} = \gamma^2 a^*_{k\mu} a_{k\mu'}, \tag{1.125}$$

where $\mu, \mu' = \pm 1$ and

$$a_{k\pm} = \mp \frac{a_{kx} \pm i a_{ky}}{\sqrt{2}}. \tag{1.126}$$

It is seen from (123)-(125) that P is the Hermitian (2 x 2) matrix. In spite of the position dependence of the mode functions in (13), the elements of (124) and (125) have the same value everywhere. This means that the polarization is the global property of classical plane waves.

In contrast to the plane waves, the field strengths of the multipole radiation can have any direction. In fact, the electric multipole radiation obey the condition $\vec{\mathcal{B}} \cdot \vec{r} = 0$, while it can have nonzero longitudinal component ($\vec{\mathcal{E}} \cdot \vec{r} \neq 0$) of the electric field strength [25]. In other words, this is ***the transversal magnetic radiation.*** In turn, the magnetic multipole field is characterized by the relations

$$\vec{\mathcal{E}} \cdot \vec{r} = 0, \qquad \vec{\mathcal{B}} \cdot \vec{r} \neq 0.$$

Hence, the polarization of either multipole radiation should be specified by the spatial anisotrophy of the field strengths rather than the transversal anisotrophy as in the case of plane waves [28, 46, 54, 88]. Thus, the polarization of the classical multipole field should be described by bilinear forms in all three components of the field strengths which leads to the Hermitian (3 x 3) matrix with the elements [28, 46]

$$P^{(E)}_{\mu\mu'}(\vec{r}) = E^*_\mu(\vec{r}) E_\mu(\vec{r}), \qquad \mu, \mu' = \pm 1, 0. \tag{1.127}$$

Here we again consider a monochromatic radiation field. Let us stress that this expression describes the spatial anisotrophy of the electric field and therefore specifies the polarization of the electric multipole radiation. In the case of magnetic multipole radiation, the spatial anisotrophy of magnetic induction can be described by the following polarization matrix [89]

$$P^{(M)}_{\mu\mu'}(\vec{r}) = B^*_\mu(\vec{r}) B_{\mu'}(\vec{r}). \tag{1.128}$$

Consider now the monochromatic multipole field with given λ and j. Exactly this field is emitted by an atomic transition. Employing (21) and (17) then gives

$$P^{(E)}_{\mu\mu'} = k^2 \sum_{mm'} V^*_{Ekjm\mu}(\vec{r}) V_{Ekjm'\mu'}(\vec{r}) a^*_{Ekjm\mu} a_{Ekjm'\mu'},$$

$$P^{(M)} = k^2 \sum_{mm'} V^*_{Ekjm\mu}(\vec{r}) V_{Ekjm'\mu'} \quad (\vec{r}) a^*_{Mkjm\mu} a_{Mkjm'\mu'}. \tag{1.129}$$

Thus, the difference between $P^{(E)}$ and $P^{(M)}$ lies in the definition of the complex field amplitudes, while the position dependence is described in the same way. Within the classical picture, these amplitudes are defined differently in terms of the source functions [25]. For example, if the source of radiation is represented by the harmonically varying current $\vec{J}(\vec{r})$ and intrinsic magnetization $\mathcal{M}(\vec{r})$, the field amplitudes for the electric radiation are determined by the integral of the function

$$\left(\vec{\mathcal{M}} + \frac{1}{ck^2}\vec{\nabla} \times \vec{J}\right)$$

over the volume of the source localization, while those for the magnetic radiation by the integrals of

$$\left(\frac{\vec{J}}{c} + \vec{\nabla} \times \vec{\mathcal{M}}\right).$$

It should be mentioned here that the quantum multipole radiation, defined in terms of the source, has been considered in [56].

In direct analogy to the case of plane waves, the diagonal elements of (129) describe the contribution of the components of the field strengths into the energy density. The off-diagonal elements give the "phase information" about the phase differences between the spatial components with different polarization. Unlike the plane waves, there are three phase differences [46, 54]

$$\Delta_{\mu\mu'}(\vec{r}) = \arg A_{E\mu}(\vec{r}) - \arg A_{E\mu'}(\vec{r}).$$

In view of the evident equality

$$\Delta_{+0} + \Delta_{0-} + \Delta_{+} = 0,$$

valid at any point $\vec{r}$, only two of the phase differences $\Delta_{\mu\mu'}$ are independent.

In contrast to (124) and (125), the polarization matrix (129) of either multipole radiation depend on the position with respect to the source (origin). Since the component $A_{\mu=0}$ vanishes at far distance much faster than the two transversal components, the polarization in the so-called far zone ($kr \gg 1$) is similar to that of the plane waves.

In addition to the Hermitian polarization matrix with complex elements, the spatial anisotrophy of the electromagnetic field can be described by an equivalent set of real Stokes parameters [14, 57].

The classical Stokes parameters of plane waves are usually defined in the linear polarization basis as follows [1]

$$\begin{aligned} s_0^{(plane)} &= |\vec{e}_x \cdot \vec{E}|^2 + |\vec{e}_y \cdot \vec{E}|^2, \\ s_1^{(plane)} &= |\vec{e}_x \cdot \vec{E}|^2 - |\vec{e}_y \cdot \vec{E}|^2, \\ s_2^{(plane)} &= 2Re[(\vec{e}_x \cdot \vec{E}^*)(\vec{e}_y \cdot \vec{E})], \\ s_3^{(plane)} &= 2Im[(\vec{e}_x \cdot \vec{E}^*)(\vec{e}_y \cdot \vec{E})]. \end{aligned} \tag{1.130}$$

In the helicity basis (16) we get

$$\begin{aligned} s_0^{(plane)} &= |E_-|^2 + |E_+|^2, \\ s_1^{(plane)} &= -2Re(E_+^* E_-), \\ s_2^{(plane)} &= 2Im(E_+^* E_-), \\ s_3^{(plane)} &= |E_-|^2 - |E_+|^2, \end{aligned} \tag{1.131}$$

where $E_\mu \equiv \vec{\chi}_\mu \cdot \vec{E}$. It is clear that the parameter $s_0^{(plane)}$ measures the relative intensity of the wave. At the same time, the expressions (130) and (131) show a rearrangement of the roles of the Stokes parameters with respect to the two bases. The parameter $s_1^{(plane)}$ in (130) gives the preponderance of the z-linear polarization over the y-linear polarization, while in (13 1) it concerns the phase difference between the two components with opposite helicities. The parameter $s_2^{(plane)}$ in (130) gives the cosine of the phase difference between the linearly polarized components, while in (131) it gives the sine of the phase difference between the circular polarized components. In turn, the parameter $s_3^{(plane)}$ in (48) specifies the sine of the phase difference in the Cartesian basis with linear polarizations, while in (13 1) it gives the preponderance of polarization with negative helicity over positive helicity.

Consider now the set of classical Stokes parameters of the monochromatic pure (λ, j)-pole radiation. Since the polarization, in this case, is described by the Hermitian (3 x 3) matrix (127), the set should consist of nine Stokes parameters because all three spatial components of the field strength contribute into polarization [46]. For definiteness, let us consider the electric-type radiation. To establish a contact with the previous result (13 1), we choose the set of Stokes parameters as follows [46]

$$\begin{aligned} s_0^{(multi)}(\vec{r}) &= \sum_\mu |E_\mu|^2, \\ s_1^{(multi)}(\vec{r}) &= -2Re(E_+^* E_0 + E_0^* E_- + E_-^* E_+), \\ s_2^{(multi)}(\vec{r}) &= 2Im(E_+^* E_0 + E_0^* E_- + E_-^* E_+), \\ s_3^{(multi)}(\vec{r}) &= |E_-|^2 - |E_+|^2, \end{aligned}$$

[1] Unfortunately, there is no uniform notations for the Stokes parameters. Our notation is that of Ref. [57].

$$
\begin{aligned}
s_4^{(multi)}(\vec{r}) &= |E_+|^2 + |E_-|^2 - 2|E_0|^2, \\
s_5^{(multi)}(\vec{r}) &= -2Re(E_+^* E_-), \\
s_6^{(multi)}(\vec{r}) &= 2Im(E_+^* E_-), \\
s_7^{(multi)}(\vec{r}) &= -2Re(E_0^* E_+), \\
s_8^{(multi)}(\vec{r}) &= 2Im(E_0^* E_+).
\end{aligned} \tag{1.132}
$$

They have very simple physical meaning. The parameter s_0 measures the contribution of all three components into the energy density. The parameters $s_{1,2,5,6,7,8}$ give the phase information about the phase differences between the circular and linear polarized components. The parameter s_3 gives the preponderance of negative helicity over positive helicity and the parameter s_4 gives the preponderance of transversal circular polarization over linear (radial) polarization. Unlike (131), the multipole Stokes parameters (132) describe the ***local properties of polarization*** due to the position dependence of the mode functions (18). It is clear that, at far distances ($kr \gg 1$) when E_0 is negligibly small in comparison with $E_\pm$, the set (132) formally coincides with (13 1) because

$$
\left\{
\begin{array}{lcll}
s_1^{(multi)} & \to & s_5^{(multi)} \propto & s_1^{(plane)} \\
s_2^{(multi)} & \to & s_6^{(multi)} \propto & s_2^{(plane)} \\
s_4^{(multi)} & \to & s_0^{(multi)} \propto & s_0^{(plane)}
\end{array}
\right. .
$$

and

$$
\left\{
\begin{array}{lcl}
s_7^{(multi)} & \to & 0 \\
s_8^{(multi)} & \to & 0
\end{array}
\right.
$$

in this limit.

1.5.2 Polarization of quantum radiation

The quantum counterpart of the polarization matrices can be constructed in direct analogy to the field quantization [90]. We have to subject the field amplitudes in (124), (125), and (129) to the Weyl-Heisenberg commutation relations (22) and (23) respectively. Thus, we get the operator matrices of polarization of the multipole radiation of the form

$$
\begin{aligned}
\hat{P}_{\mu\mu'}^{(E,n)} &= k^2 \sum_{mm'} V_{Ekjm\mu}^*(\vec{r}) V_{Ekjm'\mu'}(\vec{r}) a_{Ekjm\mu}^+ a_{Ekjm'\mu'}, \\
\hat{P}_{\mu\mu'}^{(M,n)} &= k^2 \sum_{mm'} V_{Ekjm\mu}^*(\vec{r}) V_{Ekjm'\mu'}(\vec{r}) a_{Mkjm}^+ a_{Mkjm'}.
\end{aligned} \tag{1.133}
$$

These are the Hermitian (3 x 3) matrices with the operator elements defined in terms of the normal order of the creation and annihilation photon operators.

To clarify the structure of (133) and establish a contact with previous results for the radiation phase, consider now the bare operator form of $\hat{P}^{(E,n)}$ in (133) in the case of the electric dipole radiation in a spherical cavity. The bare operator structure is provided by the limit $kr \to 0$ in the mode functions (18). This means that we consider the polarization of radiation directly near a source. Using the properties of spherical Bessel functions discussed in Subsection 3.1, we get [28, 46]

$$\hat{P}^{(E,n)}_{mm'} = \gamma_{E1} a^{+}_{Ek1m} a_{Ek1m'}, \tag{1.134}$$

where γ_{E1} is an unimportant normalization factor.

Consider now the set of Stokes operators which can be obtained by canonical quantization of (132). On the other hand, the Stokes operators should by definition represent the complete set of independent Hermitian bilinear forms in the photon operators of creation and annihilation. It is clear that such a set is represented by the generators of the $SU(3)$ sub-algebra in the Weyl-Heisenberg algebra of electric dipole photons. The nine generators have the form [46]

$$\begin{array}{lll} (a^{+}_{+}a_{+} - a^{+}_{0}a_{0}) & (a^{+}_{0}a_{0} - a^{+}_{-}a_{-}) & (a^{+}_{-}a_{-} - a^{+}_{+}a_{+}) \\ \frac{1}{2}(a^{+}_{+}a_{0} + a^{+}_{0}a_{+}) & \frac{1}{2}(a^{+}_{0}a_{-} + a^{+}_{-}a_{0}) & \frac{1}{2}(a^{+}_{-}a_{+} + a^{+}_{+}a_{-}) \\ \frac{1}{2i}(a^{+}_{+}a_{0} - a^{+}_{0}a_{-}) & \frac{1}{2i}(a^{+}_{0}a_{-} - a^{+}_{-}a_{0}) & \frac{1}{2i}(a^{+}_{-}a_{+} - a^{+}_{+}a_{-}) \end{array} \tag{1.135}$$

and only eight of them are independent. To simplify the notations we omit here the indexes E, k, and $j = 1$. To get the set of Stokes operators, we have to use the generators (135) or independent linear combinations of this generators together with the operator

$$S_0 = \sum_{m=-1}^{1} a^{+}_{m} a_{m}, \tag{1.136}$$

describing the total number of multipole photons. It seems to be natural to choose the rest of the set of Stokes operators as follows [46]

$$\begin{aligned} S_1 &= (\varepsilon_{rad} + \varepsilon^{+}_{rad}), \\ S_2 &= i(\varepsilon^{+}_{rad} - \varepsilon_{rad}), \\ S_3 &= a^{+}_{+}a_{+} - a^{+}_{-}a_{-}, \\ S_4 &= a^{+}_{+}a_{+} + a^{+}_{-}a_{-} - 2a^{+}_{0}a_{0}, \\ S_5 &= (a^{+}_{+}a_{0} + a^{+}_{0}a_{+}), \\ S_6 &= i(a^{+}_{0}a_{+} - a^{+}_{+}a_{0}), \\ S_7 &= (a^{+}_{0}a_{-} + a^{+}_{-}a_{0}), \\ S_8 &= i(a^{+}_{-}a_{0} - a^{+}_{0}a_{-}). \end{aligned} \tag{1.137}$$

Here the operator ε_{rad} is defined in (63). Thus, the operators S_1 and S_2 in (137) coincide, apart from a normalization factor, with the cosine and sine of the radiation

phase operators (82). It seems to be natural. By construction, the operators S_1 and S_2 give the "phase information" about the phase differences $\Delta_{\mu\mu'}$ defined in the previous Subsection.

Additional phase information is provided by the operators S_6-S_8 in (137). To clarify the physical meaning of the "phase-dependent" operators in (137), let us average them over the two-mode coherent state (85) at $|\alpha_+| = |\alpha_-| \equiv |\alpha|$. We get

$$\langle S_\ell \rangle = \begin{cases} 2|\alpha|^2 & \text{at } \ell = 0, 3 \\ 2|\alpha|^2 \cos \Delta_{+-} & \text{at } \ell = 1 \\ 2|\alpha|^2 \sin \Delta_{+-} & \text{at } \ell = 2 \\ 0 & \text{otherwise} \end{cases} \tag{1.138}$$

Here

$$\Delta_{+-} \equiv \arg \alpha_+ - \arg \alpha_-$$

as in Section 4.3. It is seen that the first two averages in (138) formally coincide with the conventional phase-dependent Stokes parameters defined in the helicity basis (131) [25, 57]. Hence, the Stokes operators S_1 and S_2 give the cosine and sine of the radiation phase while the operators S_5-S_8 give the cosine and sine of the phase differences $\Delta_0\pm$ between the linear and one of the circular polarizations. Since

$$[S_1, S_2] = 0, \qquad [S_1, S_0] = [S_2, S_0] = 0, \tag{1.139}$$

the operators S_1 and S_2 form the Cartan algebra in the $SU(3)$ sub-algebra (135) of the Weyl-Heisenberg algebra of multipole photons.

The remaining Stokes operators S_0, S_3, and S_4 in (133) and (134) also have simple physical meaning. In fact, S_0 describes the total number of photons (intensity), S_3 gives the preponderance of positive helicity over negative helicity, and S_4 defines the preponderance of transversal circular polarization over longitudinal linear polarization.

In spite of the formal coincidence between the Stokes parameters (138) and those obtained by quantization of (131) and further averaging over the coherent state (85), there is also a essential difference. Consider, for example, the variances of the Stokes operators S_1 and S_2 in (137):

$$\begin{aligned} \langle (\Delta S_1)^2 \rangle &= 2|\alpha|^2(2 + \cos \Delta_{+-}), \\ \langle (\Delta S_2)^2 \rangle &= 2|\alpha|^2(2 - \cos \Delta_{+-}). \end{aligned} \tag{1.140}$$

In turn, the variances of the corresponding Stokes operators describing the phase properties of plane waves of photons in the two-mode coherent state (85) have the form

$$\langle (\Delta S_1)^2 \rangle = \langle (\Delta S_2)^2 \rangle = 2|\alpha|^2. \tag{1.141}$$

Hence, the quantum fluctuations of the physical quantities, describing the phase information in the Stokes parameters, are much stronger in the case of multipole radiation in comparison with the case of plane waves of photons. Moreover, they are qualitatively different because of the phase dependence in (140).

One more difference follows from the fact that, due to the condition (139), the physical quantities described by S_1 and S_2 in (137) can be measured at once. At the same time, the operators, obtained by quantization of S_1, S_2, and S_3 in (131), form a representation of the $SU(2)$ algebra which excludes the possibility of simultaneous measurement of corresponding physical quantities.

1.53 Spatial properties of polarization

Our consideration of polarization so far have applied to the bare operator forms, corresponding to the normal-ordered polarization matrix and Stokes operators at the source location when r $\rightarrow$ 0. In reality, (133) describes the position dependent operator polarization matrix of the multipole radiation. In addition to the normal-ordered form of the operator polarization matrix (133), one can define the anti-normal form:

$$\hat{P}^{(E,an)}_{\mu\mu'} = k^2 \sum_{mm'} V^*_{Ekjm\mu}(\vec{r}) V_{Ekjm'\mu'}(\vec{r}) a_{Ekjm'} a^+_{Ekjm},$$
$$\hat{P}^{(M,an)}_{\mu\mu'} = k^2 \sum_{mm'} V^*_{Ekjm\mu}(\vec{r}) V_{Ekjm'\mu'}(\vec{r}) a_{Mkjm'\mu'} a^+_{Mkjm\mu}. \quad (1.142)$$

It is seen that, in view of the commutation relations (23), the difference

$$P^{(0)}_{\mu\mu'}(\vec{r}) = \hat{P}^{(\lambda,an)}_{\mu\mu'} - \hat{P}^{(\lambda,n)}_{\mu\mu'} = k^2 \sum_m V^*_{Ekjm\mu}(\vec{r}) V_{Ekjm\mu'}(\vec{r}) \quad (1.143)$$

defines the zero-point or vacuum oscillations of the components of the polarization of either multipole field [22, 91]. In fact, the matrix elements of $P^{(0)}$ in (143) coincide with the commutators of the type of

$$[A_{\lambda kjm\mu}, A^+_{\lambda kjm'\mu'}].$$

The trace of (143) coincides, apart from an unimportant factor, with the zero-point fluctuations of energy (29).

It is seen from (143) that the vacuum polarization matrix is independent of index λ, describing the type of the multipole field. It seems to be natural, because the vacuum properties are affected by the presence of the singular point (atom) without respect to the type of radiation which might be emitted.

A similar object can be constructed in the case of plane waves of photons. In analogy to (133) and (142), we can construct the quantum counterpart of (123) and corresponding anti-normal operator polarization matrix. Then, using the commuta-

tion relations (22) and definition (143), we get

$$P^{(0)}_{plane} = \frac{2\pi\hbar\omega_k}{V}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.144}$$

In contrast to (143), this is a diagonal matrix independent of the spatial variables. Hence, in exactly the same way as the zero-point oscillations of energy density, the vacuum fluctuations of polarization in empty space has the global nature, while those in the presence of the singular point manifest certain spatial inhomogeneity.

It is intuitively clear that the spatial properties of the vacuum noise of polarization described by (143) should be determined by the distance r from the source independently of the spherical angles θ and ϕ. In fact, all directions from the singular point should be equivalent in the absence of radiation. It is possible to say that the multipole vacuum state is degenerated with respect to the directions from the source or is invariant under rotations about an arbitrary axis, passing through the source. This degeneration is taken off by the generation of radiation of a given type λ and with given j and m, which causes the characteristic radiation pattern [25].

Consider first the polar direction when $\theta = 0$ in (18). Then, due to the known property of spherical harmonics [70]

$$Y_{j\pm1,m-\mu}(0,\phi) = \sqrt{\frac{2(j\pm1)+1}{4\pi}}\delta_{m\mu},$$

the mode functions in (19) are independent of the spherical angle ϕ. Consider pure electric multipole radiation with given j. Then, the corresponding polarization matrix in (33) takes the form

$$P^{(E,n)}_{\mu\mu'}(r,0,\phi) = k^2\mathcal{F}^*_{j\mu}(r)\mathcal{F}_{j\mu'}(r)a^+_{Ej\mu}a_{Ej\mu'}, \tag{1.145}$$

where

$$\mathcal{F}_{j\mu}(r) \equiv V_{Ejm\mu}(r,0,\phi)\delta_{m\mu}$$
$$= \sqrt{\frac{\hbar c}{2kV(2j+1)}}[\sqrt{j(2j+3)}f_{j+1}(kr)\langle 1,j+1,\mu,0|j\mu\rangle$$
$$-\sqrt{(j+1)(2j-1)}f_{j-1}(kr)\langle 1,j-1,\mu,0|j\mu\rangle].$$

It is seen that the photon operators with $|m| \geq 2$ do not contribute into the polarization of radiation in the polar direction even if $j \geq 2$. It is straightforward to calculate the elements of the vacuum polarization matrix (143) at $\theta = 0$:

$$P^{(0)}_{\mu\mu'}(r,0,\phi) = k^2\sum_{j=1}^{\infty}|\mathcal{F}^*_{j\mu}(r)|^2\delta_{\mu\mu'}. \tag{1.146}$$

Thus, the vacuum noise of polarization in the polar direction is represented by the diagonal matrix. Since the vacuum noise of polarization is supposed to be independent of the direction in the space, the matrix (143) can be put into the form (146) by a proper transformation of the reference frame spanned by the base vectors (16). We have

$$U(\vec{r})P^{(0)}(\vec{r})U^{+}(\vec{r}) = \mathcal{P}^{(0)}(r), \qquad U^{+}(\vec{r})U(\vec{r}) = 1. \tag{1.147}$$

As a result of this transformation,

$$\vec{\chi}_0(\vec{r}) \to \vec{r}/r.$$

Here

$$\mathcal{P}^{(0)}(r) = \begin{pmatrix} P_T(r) & 0 & 0 \\ 0 & P_L(r) & 0 \\ 0 & 0 & P_T(r) \end{pmatrix} \tag{1.148}$$

and

$$P_T(r) = k^2 \sum_{j=1}^{\infty} |\mathcal{F}_{j\pm}(r)|^2, \qquad P_L(r) = k^2 \sum_{j=1}^{\infty} |\mathcal{F}_{j0}(r)|^2.$$

The element P_T describes the vacuum noise of transversal (with respect to $\vec{r}$) circular polarizations with positive and negative helicity, while P_L gives the zero-point oscillations of linear polarization in the longitudinal direction (along $\vec{r}$). The explicit form of the unitary transformation (147) is

$$U_{\mu\mu'} = \frac{D_{\mu\mu'} + (1 - D_{\mu\mu})\delta_{\mu\mu'}}{\sqrt{1 + \sum_{\nu\neq\mu} |D_{\mu\nu}|^2}}. \tag{1.149}$$

Here $D_{\mu\mu'}(\vec{r})$ is expressed in terms of the elements of matrices (143) and (148) as follows

$$\begin{aligned}
D_{+0} &= \frac{1}{F_+}[P^{(0)}_{--}(P_T - P^{(0)}_{++}) + |^{(0)}P_{+-}|^2], \\
D_{+-} &= -\frac{1}{F_+}[P^{(0)}_{0-}(P_T - P^{(0)}_{++}) + P^{(0)*}_{+0}P^{(0)}_{+-}], \\
D_{0+} &= \frac{1}{F_0}[P^{(0)}_{--}(P_L - P^{(0)}_{00}) + |P^{(0)}_{0-}|^2], \\
D_{0-} &= -\frac{1}{F_0}[P^{(0)}_{+-}(P_L - P^{(0)}_{00}) + P^{(0)}_{+0}P^{(0)}_{0-}], \\
D_{-+} &= \frac{1}{F_-}[P^{(0)*}_{+0}(P_T - P^{(0)}_{--}) + P^{(0)}_{0-}P^{(0)*}_{+-}],
\end{aligned}$$

$$D_{-0} = -\frac{1}{F_-}[P^{(0)}_{++}(P_T - P^{(0)}_{--}) + |P^{(0)}_{+-}|^2],$$

where

$$F_+ = (P^{(0)*}_{+0}P^{(0)}_{--} - P^{(0)*}_{+-}P^{(0)}_{0-}),$$
$$F_0 = (P^{(0)}_{+0}P^{(0)}_{--} - P^{(0)*}_{0-}P^{(0)}_{+-}),$$
$$F_- = (P^{(0)}_{+-}P^{(0)*}_{+0} - P^{(0)}_{0-}P^{(0)}_{++}).$$

It follows from the structure of (148) that the vacuum noise of transversal (with respect to $\vec{r}$) polarization described by P_T is independent of **helicity.** At the same time, the transversal and longitudinal vacuum noise show different behaviour as functions of distance (see discussion at the end of Section 2).

To illustrate the spatial properties of the polarization of multipole radiation, consider now the normal-ordered operator polarization matrix (133) in the case of monochromatic electric-type pure j-pole radiation. Assume that the radiation field is in a single-photon state $|1_m\rangle$ with given m. Then, the average of (133) takes the form

$$\langle \hat{P}^{(E,n)}_{\mu\mu'}(\vec{r})\rangle = k^2 V^*_{Ekjm\mu}(\vec{r})V_{Ekjm\mu'}(\vec{r}).$$

It is seen that the state with given m contributes into the polarization with different μ. For the variances of the elements of the polarization matrix we get

$$\langle[\Delta\hat{P}^{(E,n)}_{\mu\mu'}(\vec{r})]^2\rangle \equiv \langle \hat{P}^{(E,n)2}_{\mu\mu'}\rangle \quad \langle \hat{P}^{(E,n)}_{\mu\mu'}\rangle^2$$
$$= k^4 V^*_{Ekjm\mu}V_{Ekjm\mu'} \sum_{m'\neq m} V^*_{Ekjm'\mu}V_{Ekjm'\mu'}.$$

It is also seen that the vacuum fluctuations of the field with m' $\neq$ ***m*** contribute into the quantum noise of polarization of the mode with given m. Similar results can be obtained for the position dependent Stokes operators obtained from (132) by canonical quantization.

1.5.4 Operator polarization matrix in the proper frame

We saw that an appropriate choice of a local reference frame leads to the diagonal representation (148) of the vacuum polarization matrix (142). The use of the unitary transformation (147) allows the operator polarization matrix (142) to be cast into the form

$$\mathcal{P}^{(E,n)}(\vec{r}) = U(\vec{r})P^{(E,n)}(\vec{r})U^+(\vec{r}), \tag{1.150}$$

where

$$\mathcal{P}^{(E,n)}_{\mu\mu'}(\vec{r}) = k^2 \mathcal{A}^+_{Ekj\mu}(\vec{r})\mathcal{A}_{Ekj\mu'}(\vec{r}), \tag{1.151}$$

and

$$\mathcal{A}_{Ekj\mu}(\vec{r}) = \sum_{\mu'=-1}^{1} U^{*}_{\mu\mu'}(\vec{r}) \sum_{m=-j}^{j} V_{Ekjm\mu'}(\vec{r}) a_{Ekjm}. \tag{1.152}$$

Similar representation can be constructed for the magnetic-type operator polarization matrix in (142) as well.

It is clear that, in view of (22), the operators (152) obey the commutation relations

$$[\mathcal{A}_{\lambda kj\mu}(\vec{r}), \mathcal{A}^{+}_{\lambda' k' j' \mu'}(\vec{r})] = \delta_{\lambda\lambda'}\delta_{kk'}\delta_{jj'}\delta_{\mu\mu'} \times \begin{cases} P_T(r) & \text{at } \mu = \pm 1 \\ P_L(r) & \text{at } \mu = 0 \end{cases} \tag{1.153}$$

Here $P_{\mu=\pm} \equiv P_T$ and $P_{\mu=0} \equiv P_L$ are the matrix elements of the diagonal vacuum polarization matrix (148). The representation of the Stokes operators in the proper frame can be constructed in the same way.

We now note that the only difference between (153) and commutation relations (23) is the presence of position-dependent factors in the right-hand side of (153). It seems to be quite tempting to introduce the normalized local operators

$$b_{\lambda kj\mu}(\vec{r}) = \frac{\mathcal{A}_{\lambda kj\mu}(\vec{r})}{\sqrt{P_\mu(\vec{r})}}, \tag{1.154}$$

which obey the standard Weyl-Heisenberg commutation relations

$$[b_{\lambda kj\mu}(\vec{r}), b^{+}_{\lambda' k' j' \mu'}(\vec{r})] = \delta_{\lambda\lambda'}\delta_{kk'}\delta_{jj'}\delta_{\mu\mu'} \tag{1.155}$$

at any point $\vec{r}$. Hence, the local transformation (152), representing the components of the operator vector potential in the proper frame, can be interpreted as a local Bogolubov canonical transformation [80], conserving the commutation relations. In fact, the equations (152) and (154) describe the transformation of global multipole photon operators $a_{\lambda kjm}$ with given $m = -j, \ldots, j$ into the local photon operators $b_{\lambda kj\mu}(\vec{r})$ with given polarization $\mu = 0, \pm 1$ at any point of the space.

Due to the form of the operator polarization matrix (142) and corresponding Stokes operators, the polarization, defined to be the spin state of photons [4, 27], is not a global property of the quantum multipole radiation. Any atomic transition emits photons with given quantum number m which yields, in view of (18), (24), and (142), the polarization of all three types depending on the distance from the atom. The structure of (152) and (154) just shows us how the photons with different m contribute into the polarization at an arbitrary point $\vec{r}$. Using the operators (154), we can construct, for example, the local bare operator representation of the polarization matrix (142) as follows

$$p^{(E,n)}_{\mu\mu'}(\vec{r}) = b^{+}_{Ekj\mu}(\vec{r}) b_{Ekj\mu'}\vec{r}), \tag{1.156}$$

as well as of the Stokes operators[2]:

$$
\begin{aligned}
S_0(\vec{r}) &= \sum_{\mu} b_{\mu}^{+}(\vec{r}) b_{\mu}(\vec{r}), \\
S_1(\vec{r}) &= \bar{\varepsilon}(\vec{r}) + \bar{\varepsilon}^{+}(\vec{r}), \\
S_2(\vec{r}) &= -i(\bar{\varepsilon}(\vec{r}) - \bar{\varepsilon}^{+}(\vec{r})), \\
S_3(\vec{r}) &= b_{+}^{+}(\vec{r}) b_{+}(\vec{r}) - b_{-}^{+}(\vec{r}) b_{-}(\vec{r}), \\
S_4(\vec{r}) &= b_{+}^{+}(\vec{r}) b_{+}(\vec{r}) + b_{-}^{+}(\vec{r}) b_{-}(\vec{r}) - 2 b_{0}^{+}(\vec{r}) b_{0}(\vec{r}), \\
S_5(\vec{r}) &= (b_{+}^{+}(\vec{r}) b_{0}(\vec{r}) + b_{0}^{+}(\vec{r}) b_{+}(\vec{r})), \\
S_6(\vec{r}) &= -i(b_{+}^{+}(\vec{r}) b_{0}(\vec{r}) - b_{0}^{+}(\vec{r}) b_{+}(\vec{r})), \\
S_7(\vec{r}) &= (b_{0}^{+}(\vec{r}) b_{-}(\vec{r}) + b_{-}^{+}(\vec{r}) b_{0}(\vec{r})), \\
S_8(\vec{r}) &= -i(b_{0}^{+}(\vec{r}) b_{-}(\vec{r}) - b_{-}^{+}(\vec{r}) b_{0}(\vec{r})),
\end{aligned}
\tag{1.157}
$$

where

$$
\bar{\varepsilon}(\vec{r}) \equiv b_{+}^{+}(\vec{r}) b_{0}(\vec{r}) + b_{0}^{+}(\vec{r}) b_{-}(\vec{r}) + b_{-}^{+}(\vec{r}) b_{+}(\vec{r}). \tag{1.158}
$$

It is seen that (157) has the operator structure and algebraic properties similar to (136)-(137). At $r \to 0$, the set (157) exactly coincides with (136)-(137). Due to the commutation relations (155), the operators (157) have the same algebraic properties as (136)-(137) at any point. In particular, we can construct the local representation of the radiation phase operators in the same way as in Section 4, using the operator (158) instead of (63). By construction, this gives us the $SU(2)$ quantum phase of spin or polarization with the properties described in Section 4.3.

Let us stress a very important difference between the representations of Stokes operators (137) and (157). If the former is valid only for the electric dipole photons, the latter describes an arbitrary multipole radiation with any λ and j. The similarity in the operator structure and quantum phase properties is caused by the same number of degrees of freedom defining the representation of the $SU(2)$ sub-algebra in the Weyl-Heisenberg algebra.

1.5.5 Resume

(i) The polarization is described by a bilinear forms in the field components corresponding to the spin states of photons. In general, the polarization is defined by nine physical parameters (operators, in the quantum picture). In the case of plane waves of photons when only two spin states are allowed, the polarization is specified by only four parameters (operators).

[2]Hereafter we omit all unimportant indexes.

(ii) The set of Stokes operators in the general case corresponds to a representation of the $SU(3)$ sub-algebra in the Weyl-Heisenberg algebra of photons. In the case of plane waves of photons, it reduces to the representation of the $SU(2)$ sub-algebra.

(iii) The $SU(2)$ quantum phase of spin (polarization) of photons described in a proper reference frame coincides with the radiation phase of electric dipole radiation discussed in Section 4.

1.6 MEASUREMENT, LOCALITY AND CAUSALITY

And then he would have lost sight of the mark he had made on the wall, where the nail was to go in, and each had to get up on the chain besides him, and see if we could find it; and we would each discover it in a different place, and he would call us all fools, one after another, and tell us to get down. And he would take the rule, and re-measure, and find that he wanted half thirty one and three-eights inches from the corner, and try to do it in his head, and go mad.

-Jerome K. Jerome: Three Men in a Boat

1.6.1 Measurement and photon localization

It was indicated in Introduction that, in spite of the fact that the photon operators refer to the radiation field ***in all space,*** it looks tempting to interpret the electronic signal registered by a photodetector as due to a photon localization in a vicinity of the sensitive area of this detector. The corresponding operational definition of localization has been done by Mandel [20] (also see [14, 15, 92]). It is based on the consideration of a plane wave of photons, being absorbed by the sensitive area s of photodetector during some finite time $\Delta\tau$. Then, it is natural to interpret this process as a measurement of photons located in a cylindrical volume $\mathcal{V} = s$. (AT) (Fig. 14).

Following [14], we introduce a photon absorption operator at the point $\vec{r}$ at time t as follows

$$\vec{a}(\vec{r},t) = \gamma \sum\nolimits'_{k,\sigma} \vec{e}_{k\sigma} a_{k\sigma} e^{i(\vec{k}\cdot\vec{r}-kct)}. \tag{1.159}$$

Here γ is the normalization factor, $a_{k\sigma}$ are the operators (21)-(22), and summation is taken over a finite set of modes to which the detector responds. The so-called *configuration space number operator* [14] is defined by the relation

$$\mathcal{N}(\mathcal{V},t) = \int_{\mathcal{V}} \vec{a}^{+}(\vec{r},t)\cdot\vec{a}(\vec{r},t)d^3r$$

$$= \gamma^2 \sum_{k,\sigma}\sum_{k',\sigma'} \vec{e}_{k\sigma}\cdot\vec{e}_{k'\sigma'} e^{-i(\vec{k}-\vec{k}')\cdot\vec{r})} e^{i(k-k')ct} a^{+}_{k\sigma} a_{k'\sigma'}, \tag{1.160}$$

where the integral is taken over the volume of photon localization. The operators (159) and (160) obey the following commutation relations [14]

$$[\mathcal{N}(\mathcal{V}, t), \mathcal{N}(\mathcal{V}', t)] = 0 \tag{1.161}$$

and

$$[\vec{a}(\vec{r}, t), \mathcal{N}(\mathcal{V}, t)] \approx \begin{cases} -\vec{a}(\vec{r}, t), & \text{if } \vec{r} \in \mathcal{V} \\ 0, & \text{otherwise.} \end{cases} \tag{1.162}$$

Let us stress, that (162) has an approximate sense.

There is a principal difference which makes difficult the direct use of the operation approach to the problem of localizing photons in the case of multipole radiation. The point is that the multipole photons are in the state with given angular momentum and therefore they have no well defined direction of propagation. In view of the wave-particle dualism, one can say that the multipole photons emitted by a point-like quantum source propagate as outgoing spherical waves. Definitely, these photons are localized initially inside the source.

Consider a model of Hertz-type experiment on emission and detection of a multipole photon in the system of two identical atoms separated by a distance d. If we assume that a photon is first emitted by the atom number one (source) and then absorbed by the atom number two (detector), it is most natural to consider the field as a superposition of outgoing and incoming spherical waves focused on the source and detector respectively (Fig. 15). This superposition should obey the boundary conditions for the real radiation field. Then, in analogy to (159) and (160), one can construct a configuration space photon number operator via the integration of the corresponding local operator over the spherical volume of radius $c\Delta\tau$, surrounding the detecting atom. Here $\Delta\tau$ is again the detection time.

Taking into account that (159) is nothing but the positive-frequency part of the vector potential (21), we can introduce the multipole photon absorption operator as follows

$$\vec{\mathbf{a}}(\vec{r}, t) = \sum_{\mu} (-1)^{\mu} \vec{\chi}_{-\mu} \sum\nolimits'_{\lambda kjm} V_{\lambda kjm\mu}(\vec{r}) e^{-ikc} a_{\lambda kjm},$$

which coincides with (24) apart from the fact that the sum is taken over the modes allowed by the selection rules. Here the mode function is defined by the equation (18) with the radial part (19) corresponding to the incoming spherical wave. Then, the configuration space multipole photon number operator takes the form

$$\mathbf{N}(\mathcal{V}, \mathrm{t}) = \int_{\mathcal{V}} \vec{\mathbf{a}}^{+}(\vec{r}, t) \cdot \vec{\mathbf{a}}(\vec{r}, t) d^3 r,$$

similar to (162). Here the volume of detection is

$$\mathcal{V} = \frac{4\pi}{3}[(c\Delta\tau)^3 - r_a^3],$$

where r_a is the atomic radius. We have to exclude the "volume of generation" occupied by the atom to avoid the divergence at $r \to 0$ in the case of outgoing and incoming spherical waves of photons. By virtue of the transformation (152), we can rewrite it as follows

$$\mathbf{N}(\mathcal{V}, t) = \int_{\mathcal{V}} \vec{\mathcal{A}}^{+}(\vec{r}, t) \cdot \vec{\mathcal{A}}(\vec{r}, t) d^3 r, \tag{1.163}$$

where the definition of the components of $\vec{\mathcal{A}}$ differs from (152) by summation over all allowed modes.

Taking into account the properties of spherical harmonics [70], Clebsch-Gordon coefficients [71], and spherical Bessel and Hankel functions [70], it is possible to show that the mode functions in (18) obey the following condition of symmetry:

$$V_{\lambda kj,-m,-\mu}(\vec{r}) = (-1)^{m+j-\mu} V^{*}_{\lambda kjm\mu}(\vec{r}).$$

Then, it can be easily seen that the commutation relation (161) is valid for the operators N as well:

$$[\mathbf{N}(\mathcal{V}, t), \mathbf{N}(\mathcal{V}', t)] = 0.$$

The commutation relation corresponding to (162) is less simple. It can be proven as an approximate one.

Thus, the picture of measurement in the atom-detector system of two identical atoms is compatible with Mandel's operational approach to the photon localization. For example, the multipole photon statistics in finite volume can be examined in the same way as in [14,201. The commutators for different t can also be constructed in analogy to [20].

Nevertheless, there is one important difference. The point is that the zero-point oscillations of multipole field are concentrated in a vicinity of the atoms, where they can strongly exceed the level calculated in the model of plane waves in empty space (see Section 2.2). If the atomic separation is large in comparison with the wavelength, then a major contribution into the vacuum noise of measurement comes from the presence of the detecting atom. At the intermediate and short distances, the vacuum noise in the vicinity of the detecting atom is increased due to the influence of the source atom via the superposed zero-point oscillations of outgoing and incoming spherical waves. Since the vacuum noise influences the precision of measurements (e.g., see [14,15, 58, 59,86]), this fact seems to be very important, especially for the experiments with trapped Ridberg atoms which are usually separated by distances corresponding to the intermediate and even near zone [32, 66].

Consider now the measurement of monochromatic plane photons by a photodetector shown in Fig. 14. At far distances, the photons are specified by a unique wave vector k. Mandel's localization of photons in the vicinity of the sensitive area σ assumes that the wave converges to σ. This means that there is a variety of directions of the wave vectors near σ (Fig. 16). This picture can be described by a proper

expansion over spherical waves. In view of the above discussion, it should lead to an increase of the vacuum noise of measurement over the level of plane waves as well.

1.6.2 Causality in the two-atom Hertz experiment

In previous Subsection, we considered the field emitted by one atom and then absorbed by another atom as a superposition of outgoing and incoming spherical waves of multipole photons. This wave picture completely eliminates an enquiry concerning the trajectory of photons between the atoms. In fact, the path of a particle in quantum mechanics is not a well-defined notion. Most that we can state about the path of a quantum particle in many cases is that it is represented by a nondifferentiable, statistically self-similar curve [93]. For example, the path of a tunneling electron and time spending in the barrier are not still defined unambiguously [94]. Moreover, recent experiments on photonic tunneling and transmission of information show the possibility of superluminal motion of photons inside an opaque barrier [95].

We now note that, according to the principles of quantum theory, not the path, but causality in the transmission of information from one object to another is important [3, 10, 11, 12, 13, 31]. In Hertzexperimentwith two atoms separated by empty space, this means that the detecting atom cannot be excited earlier than in d/c seconds after the emission of a photon by the first atom. Here d denotes the interatomic distance. Such a causality has been proven recently by Kaup and Rupasov [96]. Here we briefly discuss their proof.

In the model experiment under consideration, the field is represented by the outgoing and incoming spherical waves of photons which are specified by a continuous distribution of k or of w = ck. Assume that the two identical atoms are the two-level atoms of the type of (34) with the electric dipole transition. Due to the simple geometry of the problem (Fig. 17), it can be considered as a quasi-one-dimension integrable system [69]. The effective spatial dependence of the photon operators can be introduced with the aid of the Fourier transformation

$$c_m(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i(\omega-\omega_0)x} a_m(\omega), \tag{1.164}$$

where a,(w) $\equiv a_{E1km}$ and ω_0 is the frequency of the atomic transition. Due to (23), the operators obey the commutation relations

$$[c_m(x), c^+_{m'}(x')] = \delta_{mm'}\delta(x - x'). \tag{1.165}$$

In analogy to (34), the Hamiltonian of the system under consideration can be represented as follows

$$H = H_0 + H_{int} + \omega_0 N,$$
$$H_0 = -i \sum_m \int_{-\infty}^{\infty} c^+_m(x) \left(\frac{d}{dx} c_m(x) \right) dx,$$

$$H_{int} = \sqrt{\eta}\sum_m \sum_{f=1,2}\int_{-\infty}^{\infty}[R_{mg}^{(f)}c_m(x) + c_m^+(x)R_{gm}^{(f)}]\delta(x-x_f)dx,$$

$$N = \sum_m \left[\sum_f R_{mm}^{(f)} + \int_{-\infty}^{\infty} c_m^+(x)c_m(x)dx\right]. \quad (1.166)$$

Here $\sqrt{\eta}$ denotes the coupling constant, index $f = 1, 2$ labels the atoms and x_f is the atomic position along the x-axis (see Fig. 17), and the operator N describes the total number of excitations in the system. It is clear that

$$[N, H] = 0. \quad (1.167)$$

Using the notation of Section 3.1, we define the ground state of the system

$$|ground\rangle = |j' = 0, m' = 0\rangle_{f=1} \otimes |j' = 0, m' = 0\rangle_{f=2} \otimes |0\rangle$$

as the state of two unexcited atoms and vacuum field.

Assume that the first atom is initially excited into one of the sub-levels of the excited state with a given probability:

$$|in\rangle = \sum_m p_m R_{mg}^{(1)} \; |ground\rangle,$$

$$\sum_m |p_m|^2 = 1. \quad (1.168)$$

Then, the Schrödinger time evolution is described by the wave function

$$|\Psi(t)\rangle = e^{-iHt}|in\rangle. \quad (1.169)$$

Let us denote by $|\zeta\rangle$ the eigenstates of the Hamiltonian (166):

$$H|\zeta\rangle = E|\zeta\rangle. \quad (1.170)$$

Here, in general, ζ is a complex parameter. Then, the wave function (169) can be represented in the following way

$$|\Psi(t)\rangle = \int_{-\infty}^{\infty}\frac{d\zeta}{2\pi}L(\zeta)e^{-iEt}|in\rangle. \quad (1.171)$$

To find the eigenstates $|\zeta\rangle$ and eigenvalues E in (170), we now note that, due to the initial condition (168), the states with single excitation only are allowed in the system. These states, in general, can be chosen as follows

$$|\zeta\rangle = \sum_m \left[\sum_f \xi_m^{(f)}(\zeta)R_{mg}^{(f)} + \int_{-\infty}^{\infty} e^{i\zeta x}f_m(\zeta, x)c_m^+(x)dx\right]|ground\rangle. \quad (1.172)$$

Employing (170) then gives the following set of equations for unknown functions $f_m(\zeta, x)$ and $\xi_m^{(f)}(\zeta)$:

$$i\frac{d}{dx}[e^{i\zeta x} f_m(\zeta, x)] + Ee^{i\zeta x} f_m(\zeta, x) = \sqrt{\eta}\sum_f \xi_m^{(f)}\delta(x - x_f),$$

$$\sqrt{\eta}\int e^{i\zeta x} f_m(\zeta, x)\delta(x - x_f) = E\xi_m^{(f)}(\zeta). \qquad (1.173)$$

It is seen that the first equation in (173) gives formally a non-physical discontinuity at x $\to x_f$. To avoid this, we have to consider the finite size atoms and change $\delta(x - x_f)$ by a smooth distribution function $u(x, x_f)$, then solve the equations, and after that to put $u(x, xf) = \delta_(x - xf)$. We get

$$\begin{aligned} f_m(\zeta, x) &= \prod_{f=1,2} \frac{\zeta - i(\eta/2)sgn(x - x_f)}{\zeta + i\eta/2}, \\ \xi_m^{(1)} &= \frac{\sqrt{\eta}}{\zeta + i\eta/2} e^{i\zeta x_1}, \\ \xi_m^{(2)} &= \frac{\zeta - i\eta/2}{\zeta + i\eta/2}\frac{\sqrt{\eta}}{\zeta + i\eta/2} e^{i\zeta x_2}, \end{aligned} \qquad (1.174)$$

where

$$sgn(x - x_f) = \begin{cases} 1 & \text{atx} > xf \\ -1 & \text{a t x} < \text{x f} \\ 0 & \text{a t x} = \text{x f} \end{cases}$$

and $E = \zeta$, which is real. Thus, the function (172) is completely defined.

Since any two states (172) obey the orthogonality condition

$$\langle\zeta|\zeta'\rangle = 2\pi\delta(\zeta - \zeta'),$$

and, in view of (17 1) the initial state can be expanded in the following way

$$|in\rangle = \int \frac{d\zeta}{2\pi} L(\zeta)|\zeta\rangle,$$

for the coefficients in (172) we get

$$L(\zeta) = \langle\zeta|in\rangle = \frac{\sqrt{\eta}}{\zeta - i\eta/2}.$$

Finally, for the function (172), describing the time evolution of the system, we get

$$|\Psi(t)\rangle = \sum_m p_m \int_{-\infty}^{\infty} \frac{d\zeta}{2\pi}\frac{\sqrt{\eta}}{\zeta - i\eta/2} e^{-i\zeta t}|\zeta\rangle. \qquad (1.175)$$

This wave function can be used to calculate the evolution of any physical quantity in the system under consideration. To prove the causality, we have to calculate now the time-dependent expectation value

$$\langle R^{(2)}_{mm}\rangle_t = |p_m|^2 | \int \frac{d\zeta}{2\pi} \frac{\eta}{(\zeta + i\eta/2)^2} e^{-i\zeta(t-d/c)}|^2, \tag{1.176}$$

describing the evolution of population of corresponding sub-level of the detecting atom. It is now a straightforward matter to arrive at the following result

$$\langle R^{(2)}_{mm}\rangle = |p_m|^2 \times \begin{cases} 0 & 0 \le t \le d/c \\ \eta^2(t - d/c)^2 e^{-\sqrt{\eta}(t-d/c)} & t > d/c \end{cases} \tag{1.177}$$

(see Fig. 18). It is seen that this average shows the causal behavior.

1.6.3 Polarization measurements

Instead of discussing the well known methods of polarization measurements [87], we now turn our attention to the fluctuations in the measurement of the parameters of polarization. Following [97], consider first a fully polarized coherent plane wave in the weak quantum limit. To measure the Stokes parameters, one can use the six-port scheme shown in Fig. 19 which reflects the principal ideas of Mandel's operational approach to phase measurements [47]. Similar scheme has been analyzed in [98]. The beam whose polarization is to be measured is firstly split by the non-polarizing beam splitter BS_1. One of the output beams is sent to a polarizing beam splitter PBS_1, which defines two linearly polarized orthogonal polarization eigenmodes labeled by $i = 1, 2$. Then, the intensities I_1 and I_2 are measured by the photodetectors. The other output beam from BS_1 is further split at BS_2 with the purpose of simultaneous measuring the sine and cosine of the phase difference between the polarized components of the field. One of the output beams from BS_2 is analyzed with PBS_2, which is oriented at 45" with respect to the axes $i = 1, 2$ defined by PBS_1. The other output beam from BS_2 goes through a quarter-wave plate whose fast and slow axes are aligned along the $i = 1, 2$ directions. This beam is then analyzed at PBS_3, whose axes are aligned with PBS_2.

Consider first the measurement of the classical field. Then, for 50% non-polarizing beam splitters, the detectors $d_1, \ldots d_6$ measures the intensities

$$\begin{aligned} I_1 &= \frac{1}{2}|E_1|^2, \\ I_2 &= \frac{1}{2}|E_2|^2, \\ I_3 &= \frac{1}{4}(|E_1|^2 + |E_2|^2) + 2|E_1|^2|E_2|^2 \cos\Delta, \\ I_4 &= \frac{1}{4}(|E_1|^2 + |E_2|^2) - 2|E_1|^2|E_2|^2 \cos\Delta, \end{aligned}$$

$$I_5 = \frac{1}{4}(|E_1|^2 + |E_2|^2) + 2|E_1|^2|E_2|^2 \sin\Delta,$$
$$I_5 = \frac{1}{4}(|E_1|^2 + |E_2|^2) - 2|E_1|^2|E_2|^2 \sin\Delta. \quad (1.178)$$

Here I_ℓ denotes the intensity measured by the detector d_ℓ. From these relations the classical Stokes parameters (130) can be found as

$$s_0^{(plane)} = I_1 + I_2, \qquad s_1^{(plane)} = I_1 - I_2,$$
$$s_2^{(plane)} = I_2 - I_4, \qquad s_4^{(p\,ane)} = I_5 - I_6. \quad (1.179)$$

Here 1 $\leftrightarrow x$ and 2 $\leftrightarrow$ y. Having defined our measurement in the classical domain, we next look at the quantum case when the unused ports in the beam splitters are considered as input for vacuum fields (e.g., see [14]). These vacuum fields are indicated in Fig. 19 by v_ℓ and w_ℓ for BS and PBS, respectively.

The photon annihilation operators associated with the field at the output arms of PBS_1 are related to those at the input through

$$d_1 = ra_1 + tv_1^{(1)}, \qquad d_2 = ra_2 + tc_2^{(1)}, \quad (1.180)$$

where $r = i/\sqrt{2}$ and $t = 1/\sqrt{2}$ are the field reflection and transmission coefficients, and $v_\ell^{1)}$ describes the polarized vacuum field in the ℓ direction entering through the vacuum port of BS_1 (cf. Section 4.6). The vacuum fields $w^{(k)}$ $(k = 1, 2, 3)$ do not couple with the measured field operators since they are orthogonal to them.

In analogy to the classical definition of the angular functions

$$\sin\theta = 2\frac{\sqrt{I_1 I_2}}{I_1 + I_2}, \qquad \cos\theta = \frac{I_1 - I_2}{I_1 + I_2},$$

where θ is related to the **ellipticity** angle [87], we define the operators [97]

$$S_\theta = 2\frac{\sqrt{n_1 n_2}}{n_1 + n_2} \qquad C_\theta = \frac{n_1 - n_2}{n_1 + n_2}. \quad (1.181)$$

Here

$$n_\ell \equiv d_\ell^+ d_\ell, \qquad \ell = 1, 2$$

are the photon number operators for the fields measured at the detectors 1 and 2. Due to the Weyl-Heisenberg commutation relations for the photon operators, we get

$$[S_\theta, C_\theta] = 0, \quad (1.182)$$

so that these two quantities can be measured at once. It is also seen that

$$S_\theta^2 + C_\theta^2 = 1.$$

On the other hand, at the output ports of PBS_2 and PBS_3, the field operators are

$$\begin{aligned} d_3 &= \frac{1}{\sqrt{2}}[(tr(a_1 + a_2) + r^2(v_1^{(1)} + v_2^{(1)}) + t(v_1^{(2)} + v_2^{(2)})], \\ d_4 &= \frac{-1}{\sqrt{2}}[tr(a_1 - a_2) + r^2(v_1^{(1)} - v_2^{(1)}) + t(v_1^{(2)} - v_2^{(2)})], \\ d_5 &= \frac{1}{\sqrt{2}}[r^2(ia_1 + a_2) + tr(iv_1^{(1)} + v_2^{(1)}) + r(iv_1^{(2)} + v_2^{(2)}], \\ d_6 &= \frac{-1}{\sqrt{2}}[r^2(ia_1 - a_2) + tr(iv_1^{(1)} - v_2^{(1)}) + r(iv_1^{(2)} - v_2^{(2)}]. \end{aligned} \quad (1.183)$$

Then, in analogy to the classical functions

$$\begin{aligned} \cos\Delta &= \frac{I_3 - I_4}{\sqrt{(I_3 - I_4)^2 + (I_5 - I_6)^2}}, \\ \sin\Delta &= \frac{I_5 - I_6}{\sqrt{(I_3 - I_4)^2 + (I_5 - I_6)^2}}, \end{aligned}$$

we get the operators

$$\begin{aligned} C_\Delta &= \frac{n_3 - n_4}{\sqrt{(n_3 - n_4)^2 + (n_5 - n_6)^2}}, \\ S_\Delta &= \frac{n_5 - n_6}{\sqrt{(n_3 - n_4)^2 + (n_5 - n_6)^2}}, \end{aligned} \quad (1.184)$$

such that

$$[C_\Delta, S_\Delta] = 0.$$

Here n_ℓ denotes the number operators constructed from the photon operators (183).

It is now a straightforward matter to calculate the variances of the operators (184)

On the other hand, at the output ports of PBS_2 and PBS_3, the field operators are

$$\begin{aligned} d_3 &= \frac{1}{\sqrt{2}}[(tr(a_1 + a_2) + r^2(v_1^{(1)} + v_2^{(1)}) + t(v_1^{(2)} + v_2^{(2)})], \\ d_4 &= \frac{-1}{\sqrt{2}}[tr(a_1 - a_2) + r^2(v_1^{(1)} - v_2^{(1)}) + t(v_1^{(2)} - v_2^{(2)})], \\ d_5 &= \frac{1}{\sqrt{2}}[r^2(ia_1 + a_2) + tr(iv_1^{(1)} + v_2^{(1)}) + r(iv_1^{(2)} + v_2^{(2)}], \\ d_6 &= \frac{-1}{\sqrt{2}}[r^2(ia_1 - a_2) + tr(iv_1^{(1)} - v_2^{(1)}) + r(iv_1^{(2)} - v_2^{(2)}]. \end{aligned} \quad (1.183)$$

Then, in analogy to the classical functions

$$\begin{aligned} \cos\Delta &= \frac{I_3 - I_4}{\sqrt{(I_3 - I_4)^2 + (I_5 - I_6)^2}}, \\ \sin A &= \frac{I_5 - I_6}{\sqrt{(I_3 - I_4)^2 + (I_5 - I_6)^2}}, \end{aligned}$$

we get the operators

$$\begin{aligned} C_\Delta &= \frac{n_3 - n_4}{\sqrt{(n_3 - n_4)^2 + (n_5 - n_6)^2}}, \\ S_\Delta &= \frac{n_5 - n_6}{\sqrt{(n_3 - n_4)^2 + (n_5 - n_6)^2}}, \end{aligned} \quad (1.184)$$

such that

$$[C_\Delta, S_\Delta] = 0.$$

Here n_ℓ denotes the number operators constructed from the photon operators (183).

It is now a straightforward matter to calculate the variances of the operators (18 1) and (184) [97]. In particular, it is possible to show that the quantum fluctuations imply the uncertainty relations

$$\begin{aligned} \langle(\Delta S_2^{(plane)})^2\rangle\langle(\Delta C_\Delta)^2\rangle &\geq \frac{1}{4}\langle S_\Delta\rangle, \\ \langle(\Delta S_2^{(plane)})^2\rangle\langle(\Delta S_\Delta)^2\rangle &\geq \frac{1}{4}\langle C_\Delta\rangle, \end{aligned} \quad (1.185)$$

similar, in some sense, to the Susskind-Glogower uncertainty relations [41] and those of the operational definition of quantum phase [47]. Here $S_2^{(plane)}$ is the Stokes operator obtained by the quantization of the Stokes parameter $s_2^{(plane)}$ in (179).

1.6.4 Nondemolition polarization measurement

In previous Subsection, we discussed the polarization measurement through the use of the detection of the field variables done by photocounting techniques, which are field destructive. As a result, successive measurements of the field variables yield different results. It seems to be tempting to use the measurement schemes avoiding back action of the detecting device on the detected observable. Such a measurement is usually called a quantum nondemolition measurement [99]. Here, following our results [100], we discuss the possibility of quantum nondemolition measurement of polarization of the electromagnetic field via the Aharonov-Bohm effect [101].

First of all, we note that the standard photodetection is a *local* measurement of the field variables (intensities). At the same time, the Aharonov-Bohm effect represents a *topological* measurement referred to the properties of vector potential along some loop. In the usual form, the Aharonov-Bohm effect deals with static or slowly time-varying magnetic fields [101]. The effect consists in the appearance of a persistent current in a metallic loop over which the magnetic flux passes. This current is a periodic function of magnetic flux with the period of flux quantum hc/e. Besides that, certain resistance oscillations in the loop incorporated into an external circuit with the same period can occur.

An important case of varying magnetic field has been considered recently by Aronov *et* al [102] under the assumption that the space dependent time-varying electromagnetic field produces static electron energy minibands in the loop. These minibands have been suggested to appear due to electron motion in a time-averaged electrostatic potential periodic with coordinate along the loop circumference, produced by the square of time-varying electric field [103]. However, in the quantum case, an electron reflection form an oscillating potential causes time-dependent phase shifts resulting in an effective chaotization of the phase of electron wave function, except at energy multiples of $\hbar\omega$, where ω is the field frequency.

In our papers [100], we considered the case of optical frequencies w > $\Delta E/\hbar$, where ΔE is the width of the electron conduction band of the metal. Under this condition, the elastic scattering of electrons is prohibited if the separation between the conduction and higher non-occupied bands of a metal is larger than $\hbar\omega$. In this case, the magnetic component of electromagnetic field represents the main source of the electron wave function phase shift. The effect of oscillating magnetic field results in the modulation of the electron transmission amplitude between the parts of the loop. Due to the quantum interference of electron waves in oscillating potential, the dependence of the loop resistance of the time-varying field amplitude manifests a non-monotone character.

The geometry of the experiment assumes that the magnetic field oscillates along the axis orthogonal to the plane of the loop circumference and passing through its center. An example is provided by a small metallic ring surrounding an optical fiber. In this case, the largest contribution to the conductance oscillations comes from the TE_{01} mode of the fiber field [100] (about the fiber modes, see Ref. [104]). Definitely, such a measurement does not perturb the quantum state of the fiber mode.

Another example is provided by the magnetic dipole radiation, when a longitudinal oscillating magnetic field can be observed, at least in some vicinity of the source. In the case of radio-band frequencies, this vicinity seems to be extended enough to make a macroscopic measurement. As a powerful localized source of such a radiation the radio-band Dicke superradiance [105] can be used. The geometry of the experiment is simplified, in this case, by a characteristic sharp radiation pattern of superradiance [64].

To make necessary estimations, consider a one-dimensional loop in a tight-binding approximation with two transmittance amplitudes t_1 and t_2 at the points A and B, connecting two parts of the ring (see Fig. 20). It is supposed that $t_1, t_2 \ll t_0$, the hopping amplitude between the nearest points inside the upper and lower parts of the ring. The system under consideration is described by the Hamiltonian [100]

$$\begin{aligned} H &= -t_0 \sum_n (a_n^+ a_{n+1} + b_n^+ b_{n+1}) + H.c. + H_{int}, \\ H_{int} &= -t_1 a_{n_1}^+ b_{n_1} e^{i\alpha_1} - t_2 a_{n_2}^+ b_{n_2} e^{i\alpha_2} + H.c., \end{aligned} \tag{1.186}$$

where a, b are the electron (not photon) annihilation operators. The phases of transmission amplitudes at the contraction points n_i are

$$\alpha_i = \alpha_i^0 + A_i \sin(\omega t + \delta_i),$$

where α_i^0 accounts for the effect of a static magnetic field applied perpendicular to the plane of the ring,

$$\alpha_1^0 - \alpha_2^0 = \frac{2\pi\Phi_S}{\Phi_0}, \qquad \Phi_s = \int \vec{B} \cdot d\vec{S}, \qquad \Phi_0 = \frac{hc}{e},$$

while A_i are the amplitudes of high-frequency field at corresponding points.

The model with the Hamiltonian (186) can be solved exactly [100]. It is a straightforward matter to arrive at the following relation for the current

$$\begin{aligned} J = \int_0^\pi \frac{dk}{2\pi} W_0 \left[\frac{W_k}{W_0 + W_k} + \frac{W_{-k}}{W_0 + W_{-k}} \right] \\ \times \left[f_0\left(\epsilon_k - \frac{eV}{2}\right) - f_0\left(\epsilon_k + \frac{eV}{2}\right) \right]. \end{aligned} \tag{1.187}$$

Here $f_0(\cdot)$ denotes the equilibrium distribution function of electrons, $\epsilon_k = -2t_0 \cos k$, $W_{\pm k}$ is the forward $(+k)$ or backward $(-k)$ scattering probability, and V is the voltage. Then, the conductance G can be found from (187) as follows

$$G = \frac{dJ}{dV}. \tag{1.188}$$

It follows from (187) that the dependence of G on the phase α and on the electromagnetic field amplitude leads to two different effects. First, the oscillatory dependence

$G(\Phi_S)$ is the standard mesoscopic interference effect similar to that in static electron interferometer [101]. Another type of oscillating dependence $G(A)$ is completely caused by the time-varying field. The dependence of the conductance on the field intensity P is shown in Fig. 21. The observation of such oscillations requires low enough temperatures at which phase-breaking length of electron scattering ℓ_ϕ exceeds the circumference of the ring L. For the loop size of the order of $L = 1\mu m$, which is technically reasonable, the condition $\ell_\phi > L$ is valid at $T \leq 1K$.

At $L = 1\mu m$, the measurement can be done at the magnetic field $B \sim 10^{-7}T$, which corresponds to the oscillating power $P \sim 10^{-3}w$ [100]. This seems to be not very high. Estimated in a different way as a minimum number of optical photons transmitted through the loop, the field should contain $n \sim 1/\alpha \sim 137$ photons, where $\alpha \equiv e^2/\hbar c$ is the fine structure constant. This estimation corresponds either to an optical soliton propagating through the fiber [104] or to the superradiant pulse in the radio-band super-radiance [105].

Thus, the linearly polarized longitudinal component of electromagnetic radiation arising in corresponding geometry (fiber or localized source) can be measured in nondemolition way with the aid of the Aharonov-Bohm effect at optical frequencies.

1.6.5 Resume

(i)Localization of multipole photons at generation and absorption by an atom, described in terms of outgoing or incoming waves of photons, is compatible with Mandel's conception of localization.

(ii) The photon localization at detection leads to a strong increase of the vacuum noise in a certain vicinity of a detector (atom or active area of photodetector).

(iii) The two-atom scheme of the Hertz experiment with multipole photons, in which the radiation field is described by a superposition of outgoing and incoming waves focused on the emitting and detecting atoms respectively, obey the causality principle in spite of the fact that the path of detecting photons is indefinite.

(iv) The quantum fluctuations of polarization can be measured in the multi-port six-detector scheme similar to that proposed by Mandel for the phase measurements.

(v) The linear polarization of longitudinal component of electromagnetic field caused by a characteristic geometry of the system (cylindrical geometry in an optical fiber or spherical in the case of field generated by a local source) can be measured in a nondemolition way with the aid of the Aharonov-Bohm effect.

1.7 CONCLUSION

Every problem becomes very childish when once it is explained to you.
-Sir Arthur Conan Doyl: The Dancing Men

In this paper we have reviewed some recent results concerning the quantum multipole radiation. Although the representation of quantum electromagnetic radiation in terms of spherical waves of photons known since the first edition in 1936 of the famous book by Heitler on quantum theory of radiation [2], where this subject is discussed in Appendix, this representation is not a widespread one. The spherical waves of photons are considered in very few advanced monographs on quantum optics (e.g., see [26]). The brilliant encyclopaedic monographs [14, 15] just touch upon the subject.

At the same time, the quantum multipole radiation is precisely what the atomic transitions between the states with given angular momenta emit. The states of multipole (spherical) photons are specified by given angular momentum and projection of the angular momentum, while the spin state (polarization) is changed in space-time. On the contrary, the common representation of plane waves of photons is specified by a given linear momentum and polarization everywhere. This difference reflects the boundary conditions used in the canonical quantization of the free electromagnetic field [2].

It was shown in Section 1 that the clear-cut distinction between the symmetry properties of plane and spherical waves leads to a qualitative difference in the zero-point oscillations. While the former are homogeneous in the space (along the direction of propagation), the latter concentrate in a certain vicinity of a local source (atom) where they may exceed the level calculated in the model of plane waves. Although the result is simple in itself, to our knowledge, it was indicated for the first time in [46] and then discussed in [22, 29]. Since the zero-point oscillations define the quantum limit of precision of measurements of corresponding physical quantity [14, 15,991, this result seems to be very important. The estimations based on the results of Section 2.2 show that the zero-point oscillations exceed the conventional level provided by the representation of plane waves at the distances $r \leq 0.2\lambda$, where λ is the wave length. Within the standard classification, it corresponds to the so-called intermediate zone. Let us stress that this distance is of the order of the typical interatomic separation in experiments with trapped Ridberg atoms [32].

One of the major trends of current research is the study of transmission of "information" between the atom and photons in the process of emission and absorption. In particular, the conservation of angular momentum provides the transmission of the "quantum phase information" in the atom-field system. The atomic quantum phase can be constructed as the $SU(2)$ phase of the angular momentum of the excited atomic state (Section 3). It is shown that this phase has very close connection with the EPR paradox and entangled states in general. Via the integrals of motion, it is mapped into the Hilbert space of multipole photons (Section 4.1). This mapping is adequately described by the dual representation of multipole photons, constructed in [46] (see Section 4.2). Instead of the quantum number m, corresponding to the projection of angular momentum, these photons are specified by the quantum phase index. The spectrum of corresponding quantum (radiation) phase is discrete and lies in the interval $(0, 2\pi)$. In the classical limit provided by the high-intensity coherent multipole radiation, the eigenvalues of the radiation phase are distributed uniformly

over this interval. By construction, the radiation phase is complementary to Mandel's operational phase [47]. It defines the quantum phase in terms of what *can* be *generated* by a quantum local source. In the quantum limit of weak intensity, the behaviour of radiation and Pegg-Bamett [45] quantum phases is completely different. It is caused by the specific truncation procedure used within the Pegg-Barnett approach, which leads to an effective violation of the Weyl-Heisenberg algebra of photons.

The radiation phase is closely related to the polarization of the multipole field. In contrast to the plane waves, the polarization of multipole radiation is specified by three orthogonal directions of the field oscillations or by three allowed spin states of photons [28]. Therefore, the polarization is described by the (3 x 3) Hermitian matrix of polarization (operator matrix, in the quantum case) (Sections 5.1 and 5.2). This general form of the polarization matrix gives in the limit the standard case of (2 x 2) matrix in the representation of plane waves. The set of multipole Stokes operators corresponds to a representation of the $SU(3)$ sub-algebra in the Weyl-Heisenberg algebra of photons. The Cat-tan algebra of this $SU(3)$ sub-algebra corresponds to the representation of the radiation phase in the case of the angular momentum $j = 1$. Thus, the polarization of multipole radiation has a certain inherent quantum phase which, in some important cases, can be interpreted as the phase difference between the components with different polarization (Section 5.2). The quantum properties of polarization can be measured with the aid of the multi-port scheme proposed in [97] (Section 6.3).

The polarization and quantum phase properties of multipole photons change with the distance from the source. This dependence can be adequately described with the aid of the local representation of the photon operators proposed in [91] and discussed in Section 5.4. In this representation, the photon operators of creation and annihilation correspond to the states with given spin (polarization) at any point. This representation may be useful in the quantum near-field optics. As we know, so far the near-field optics is mainly based on the classical picture of the field (e.g., see [106]).

The local representation of multipole photons is compatible with the Mandel's operational definition of photon localization [20]. In addition to the localization at photodetection, it permits us to describe a complete Hertz type experiment with two identical atoms used as the emitter and detector (Section 6.1). In spite of the fact that the photon path is undefined from the quantum mechanical point of view, the measurement process in such a system obeys the causality principle (Section 6.2). The two-atom Hertz experiment can be realized for the trapped Ridberg atoms. Let us stress that such a measurement is closely connected with the problem of atomic entanglement discussed in [76].

The standard measurement of different properties of quantum electromagnetic radiation is based on the photodetection, which is field destructive. Following our consideration of the possibility of the Aharonov-Bohm effect at optical frequencies [100], we propose here a new nondemolition method of polarization measurement in which the linearly polarized longitudinal mode of the field is detected without any perturbation of its quantum state (Section 6.4). The estimation of physical conditions

shows that such a measurement can be done either for the photons propagating through the fiber, or for the superradiant photons in radio-band frequencies.

In this review, we relied to a large extent on our own results. We believe, however, that this paper covers a number of topics important for quantum optics, and that the reported results may stimulate an interest in further investigations. Recent success in the field of quantum optics has convicingly shown that developments in optics have very often had a direct influence on other fields, both pure and applied.

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Table 1.1 **List of eigenvalues (81). Here $\ell = 0, \pm 1$.**

$n = 1$	$\epsilon^{(1)} = 1$	$\varphi^{(1)} = 2\ell\pi/3$
$n = 2$	$\epsilon^{(2)} = 2$	$\varphi^{(2)} = 2\ell\pi/3$
	$\epsilon^{(2)} = 1$	$\varphi^{(2)} = (2\ell + 1)\pi/3$
$n = 3$	$\epsilon^{(3)} = 3$	$\varphi^{(3)} = 2\ell\pi/3$
	$\epsilon^{(3)} = \sqrt{3}$	$\phi^{(3)} = \pm(\pi/2 + \ell\pi/3)$
$n = 4$	$\epsilon^{(4)} = 4$	$\varphi^{(4)} = 2\ell\pi/3$
	$\epsilon^{(4)} = \sqrt{7}$	$\varphi^{(4)} = \pm \tan^{-1}(\sqrt{3}/5) + 2\ell\pi/3$
	$\epsilon^{(4)} = 2$	$\varphi^{(4)} = (2\ell + 1)\pi/3$
	$\epsilon^{(4)} = 1$	$\varphi^{(4)} = 2\ell\pi/3$
$n = 5$	$\epsilon^{(5)} = 5$	$\varphi^{(5)} = 2\ell\pi/3$
	$\epsilon^{(5)} = \sqrt{13}$	$\varphi^{(5)} = \tan^{-1}(\sqrt{3}/7) \pm 2\ell\pi/3$
		$\varphi^{(5)} = \pm \tan^{-1}(\sqrt{3}/7) + 2\ell\pi/3$
	$\epsilon^{(5)} = \sqrt{7}$	$\varphi^{(5)} = \pm \tan^{-1}(\sqrt{3}/5) + \pi + 2\ell\pi/3$
	$\epsilon^{(5)} = 2$	$\varphi^{(5)} = 2\ell\pi/3$
	$\epsilon^{(5)} = 1$	$\varphi^{(5)} = (2\ell = 1)\pi/3$

Fig. 1.1 Contribution into the zero-point oscillations (29) from the terms with $j = 1$ and in the case of an ideal spherical cavity as a function of $x = kr$. (b) The level (28) is shown by the straight line.

Fig. 1.2 Contribution into (29) from the term $j = 1$ for outgoing spherical waves outside the atom with radius $r_a = 1$ in arbitrary units. The straight line shows the level of $W_k^{(plane)}$.

Fig. 1.3 Transmission of information from atom to detector as propagation of the grin of Cheshire Cat. In the upper picture, the atom keeps information about excited level (Cat's green), In the lower picture, the atom hands this information to a photon, propagating to a detector.

Fig. 1.4 Scheme of transitions between the triple degenerated excited state $j = 1$, $m = 0, \pm 1)$ and ground state $j' = 0$, m' $= 0$) in a two-level atom with the dipole transition.

Fig. 1.5 The structure of eigenvalues (80), (81) corresponding to the phase states with $n = 1, \ldots, 5$ photons. The bold lines correspond to the double-degenerated eigenstates.

Fig. 1.6 The dependence of variances (90) on $|\alpha_+|$ at fixed $|\alpha_-| = 0.275$ and $\Delta_{+-} = \pi/6$.

Fig. 1.7 Probability (93) versus **A+-.**

Fig. 1.8 Lower estimate of the probability to have the radiation phase $\varphi = 2\pi/3$ as a function of $|\alpha|^2$ at **A+- =** $2\pi/3$.

Fig. 1.9 Rabi oscillations of the variance of the field cosine operator as a function of scaled time $t_s = gt/(2\pi\sqrt{n_+ + n_-})$ at $n_\pm = 25$ and g = 1. Graphs from top to bottom correspond to the relative phase **A+-. =** 0^o, 45^o, 75^o, 90^o, respectively.

Fig. 1.10 Variance of cosine. The lower curve is for the Pegg-Barnett cosine, the upper curve is for the radiation phase cosine. Both curves are drawn at **A+- = 0** and I- = 0.275.

Fig. 1.11 Outline of the scheme of two-port interferometer.

Fig. 1.12 Variance of the cosine of radiation phase (119) in the two-mode coherent state at $|\alpha_+|^2 = |\alpha_-|^2 = |\alpha|^2$ versus $|\alpha|^2$.

Fig. 1.13 Variance of the cosine of radiation phase (119) in the two-mode coherent state as a function of $|\alpha_+|^2$ at fixed $|\alpha_-|^2 = 0.1$ respectively.

Fig. 1.14 Photodetection and region of localization.

Fig. 1.15 The scheme of two-atom Hertz experiment.

Fig. 1.16 Absorption of radiation by photodetector.

Fig. 1.17 Effective geometry of the model. The z-axis corresponds to the quasi-one-dimensional time-dependent wave function.

Fig. 1.18 Time dependence of the mean population of the excited level of the detecting atom.

Fig. 1.19 The experimental setup for measurement of the Stokes parameters.

Fig. 1.20 The scheme of one-dimensional Aharonov-Bohm loop, surrounding the direction of propagation of a longitudinal mode and weakly coupled at points *A* and *B* with the external leads L_1 and L_2 ((a) and (b)). (d) The model of an ac normal-metal interferometer. R_1 and R_2 are the thermal reservoirs held at voltages $\pm V/2$ respectively.

Fig. 1.21 dc conductance of the loop versus square root of ac power: solid line, $A_1 : A_2 = 1 : 1$; dotted line, $A_1 : A_2 = 1 : 2$. Change in the conductance is normalized with respect to static conductance oscillation amplitude.

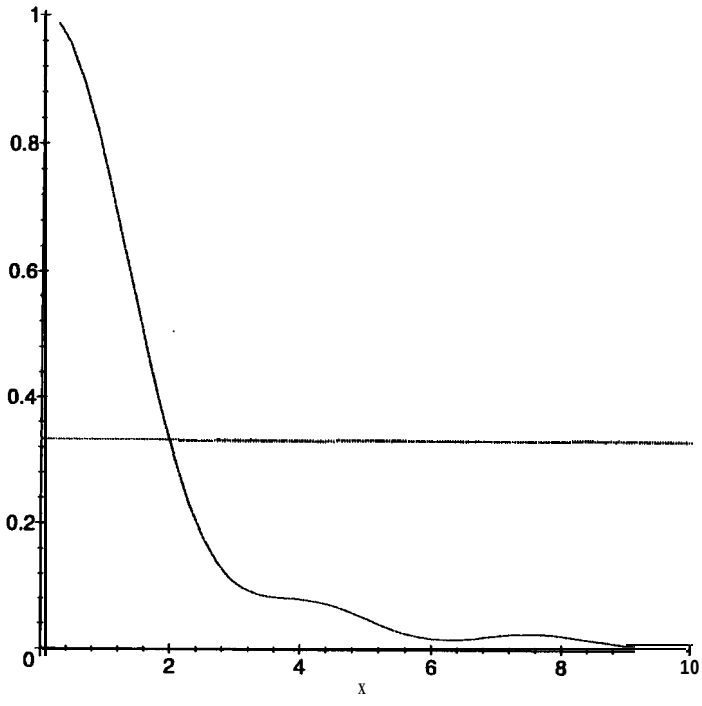

Fig. 1.
Shumovsky

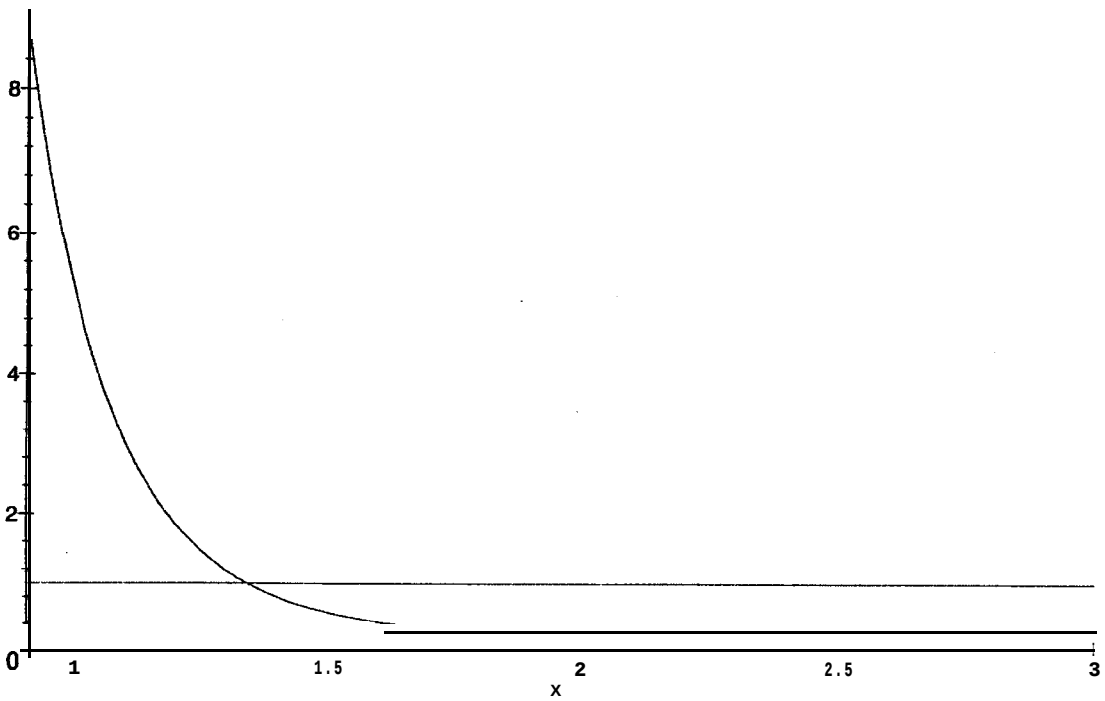

Fig 2
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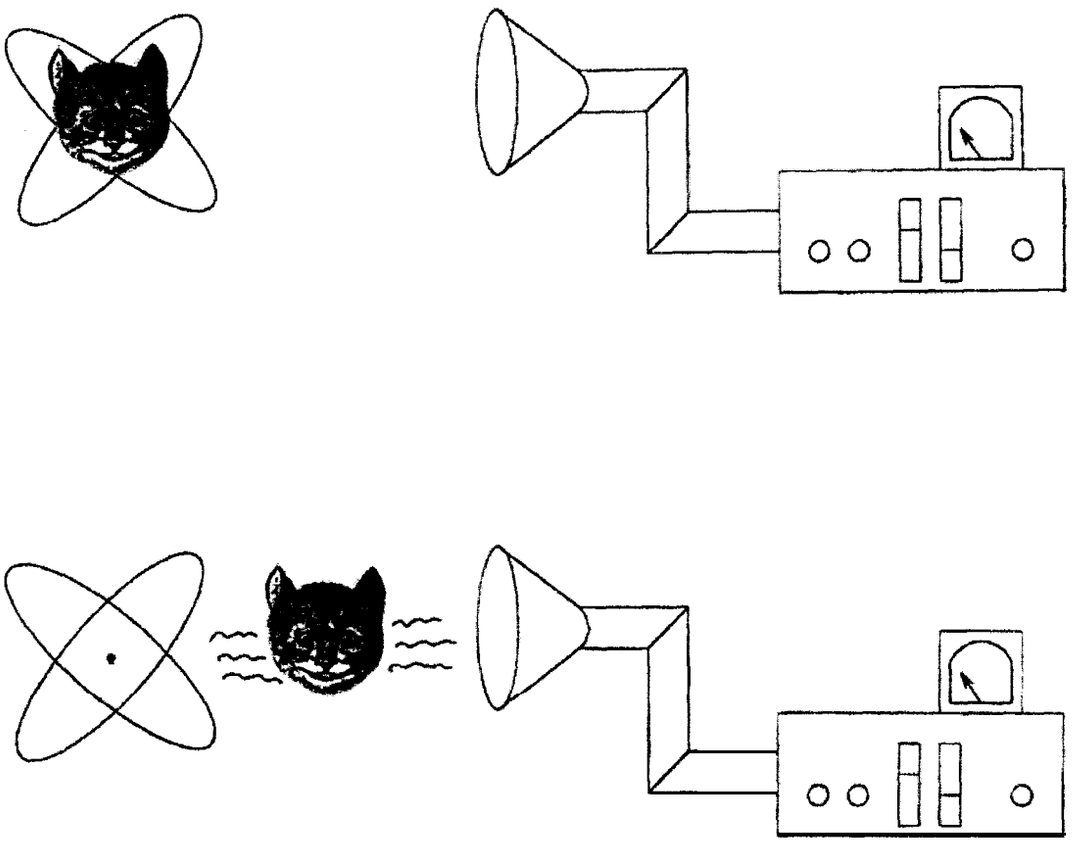
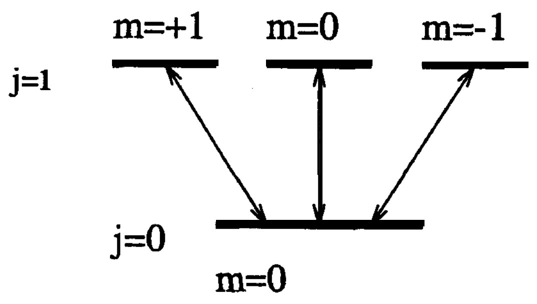


Fig. 3
S. Horvitz



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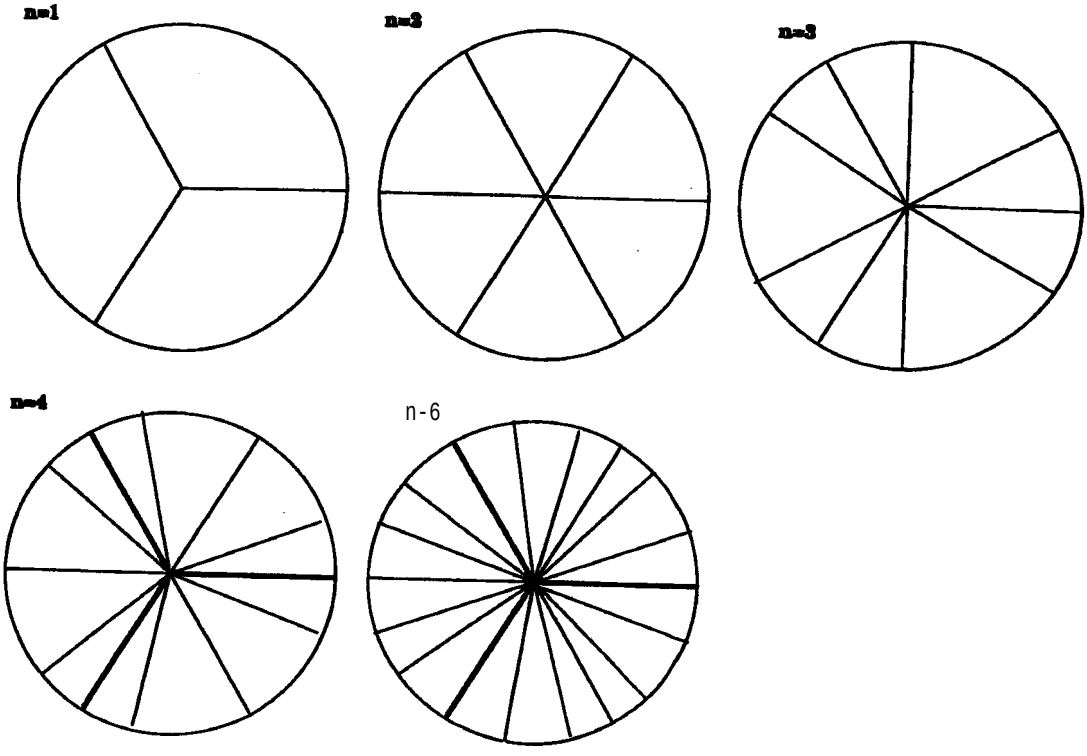

Fig. 5
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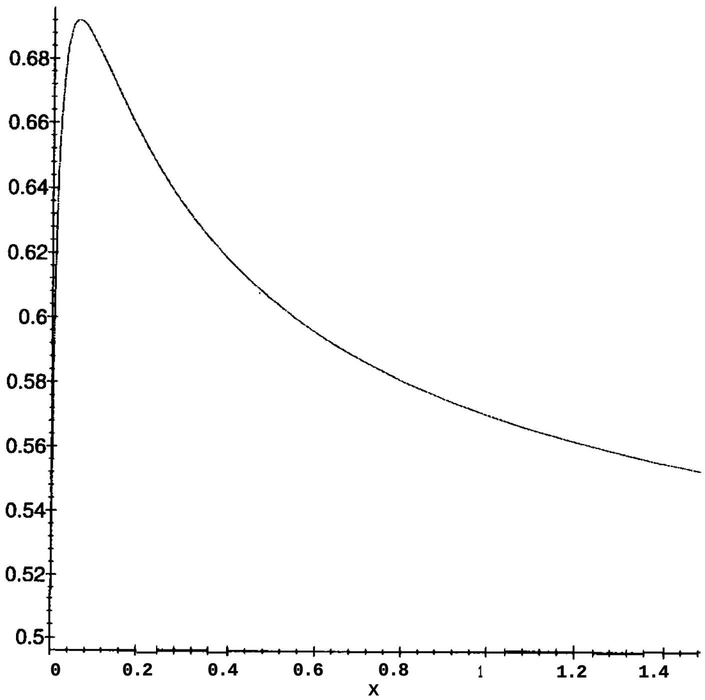
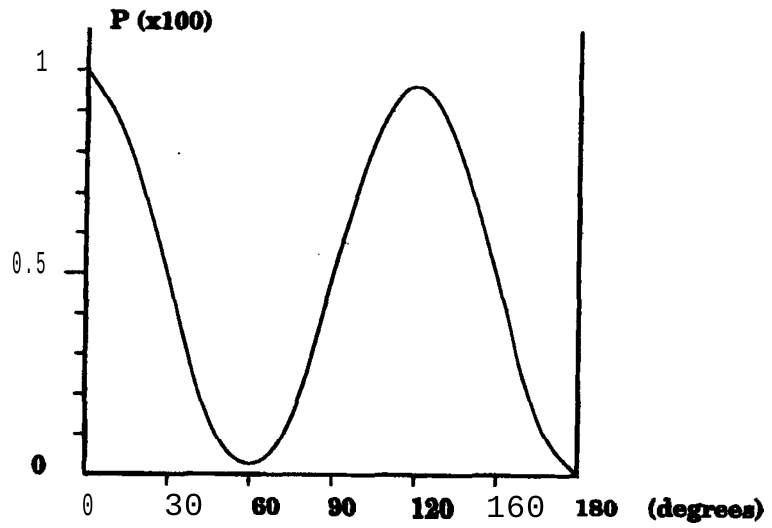


Fig 6
Shumovsky



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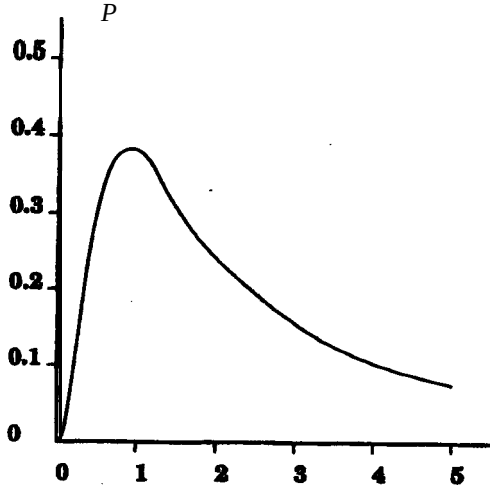
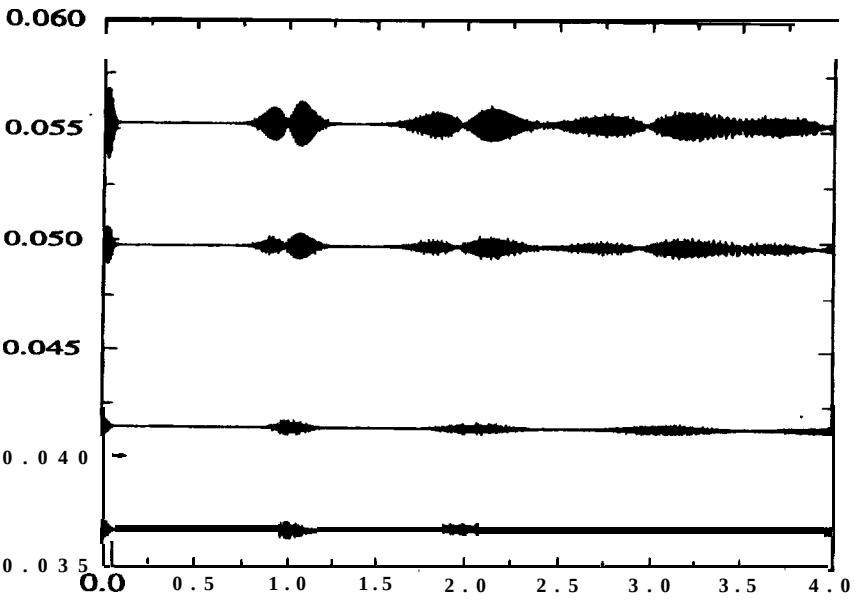


Fig 8



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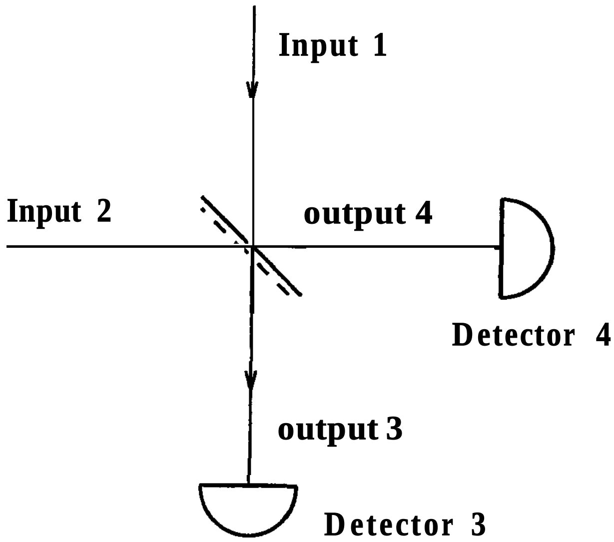

Fig. 11
Shumovsky

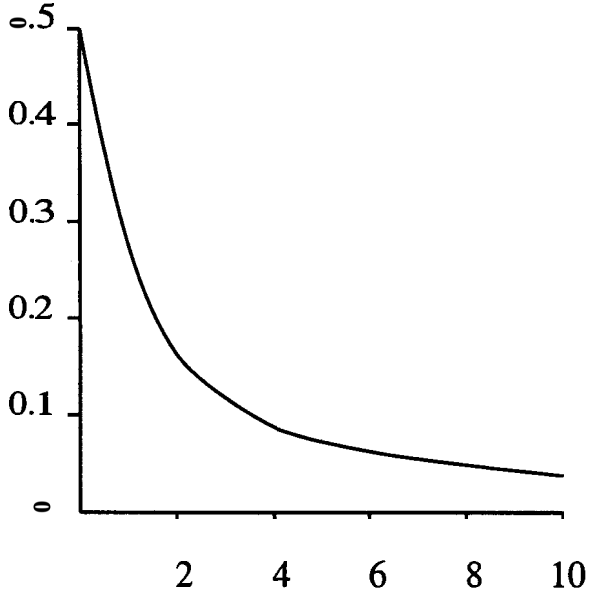

Fig. 12
Sheet 11

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0.1
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0.3
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2
4
6
8
10

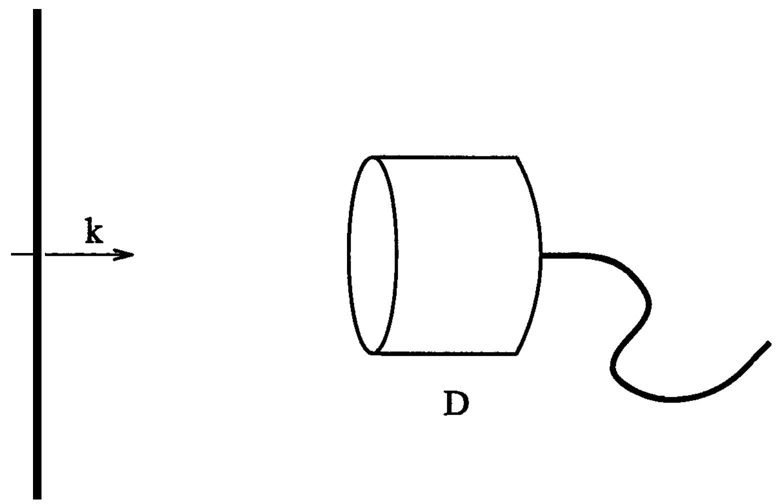

Fig. 14
Shumovsky

Fig. 15

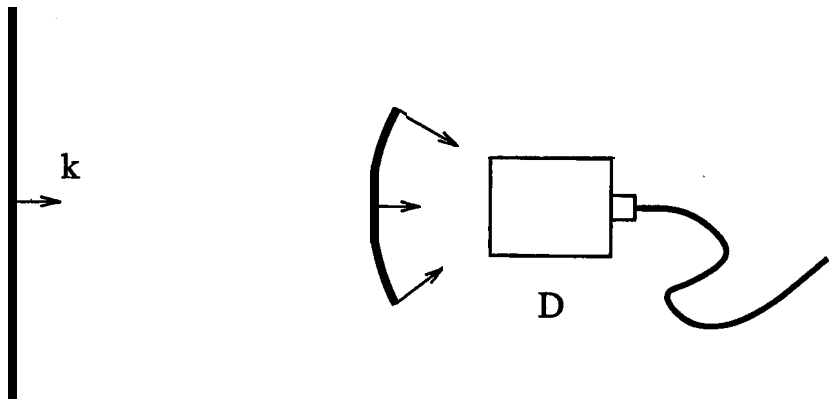

Fig. 16
Shumovsky

p
x

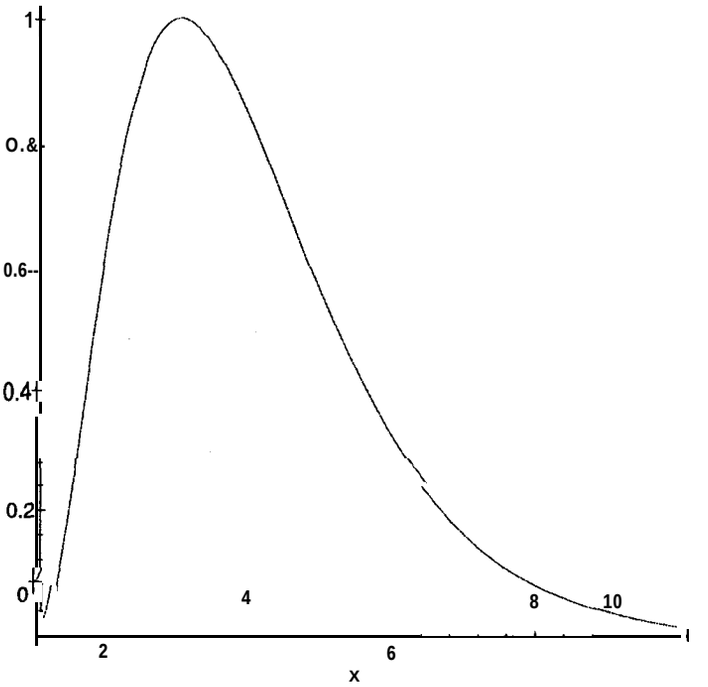

Fig 18
Skewness

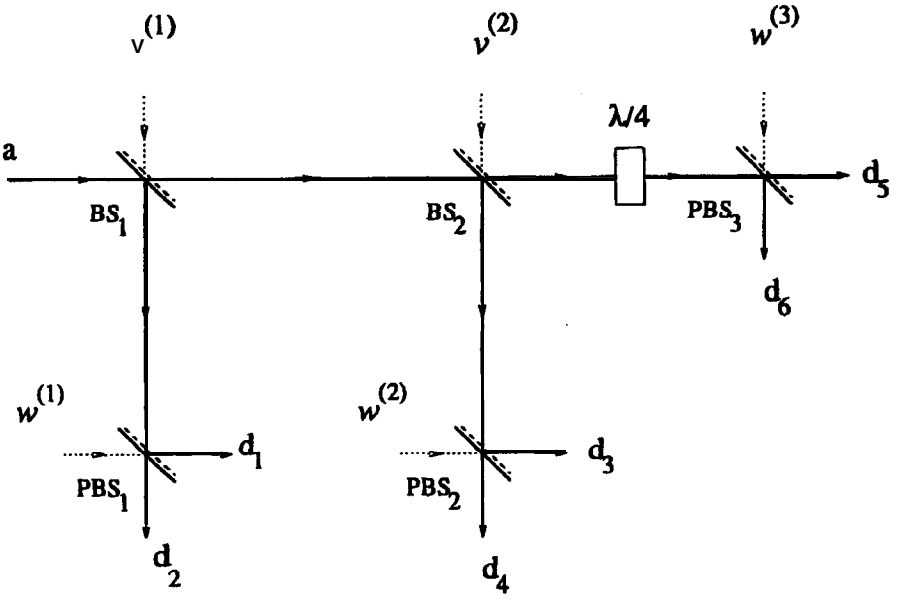

Fig. 19

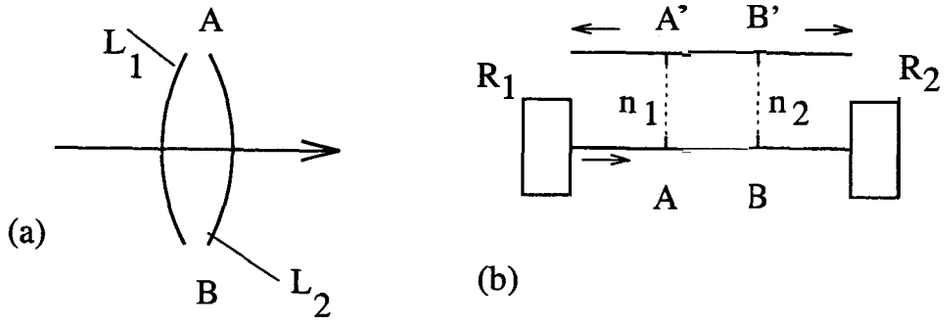

Fig. 20
Shirasawa

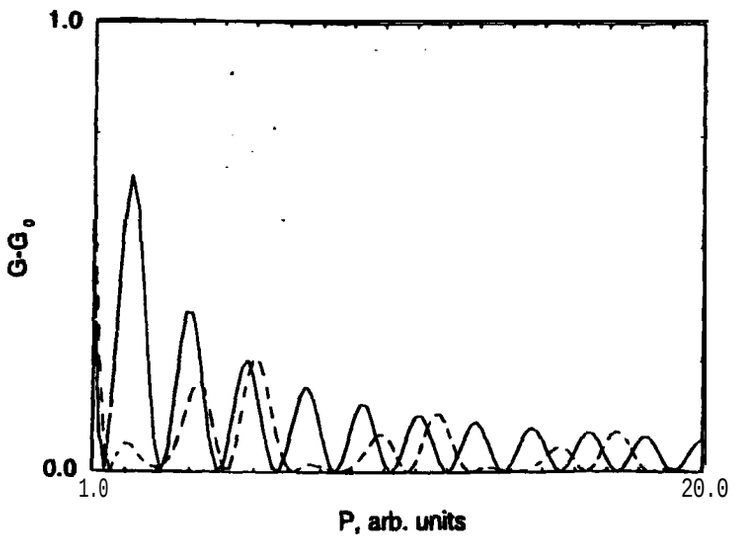

Fig. 21
Shamovsky